C*-algebras of certain non-minimal homeomorphisms on a Cantor set

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Minimal Cantor System

Let X be a Cantor set, and let $\sigma : X \to X$ be a minimal homeomorphism. Let $y \in X$. Consider

$$A = \mathrm{C}(X) \rtimes_{\sigma} \mathbb{Z}$$

and

$$A_y = \mathrm{C}^*\{f, gu : \ f, g \in \mathrm{C}(X), \ g(y) = 0\} \subseteq A.$$

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Theorem

A is a simple $A\mathbb{T}$-algebra, and A_y is a simple AF-algebra. Moreover, $K_0(A) \cong K_0(A_y)$ as order-unit groups.

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2. For every vertex v, the set of edges $r^{-1}(v)$ which end at v form a totally ordered set. It induces a lexicographical order on the set of infinite paths, i.e.,

$$(\xi_1, \xi_1, ...) > (\eta_1, \eta_2, ...)$$

if an only if there is N such that

$$\xi_n = \eta_n, \quad \forall n > N \quad \text{and} \quad \xi_N > \eta_N.$$

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   $$\xi_n = \eta_n, \quad \forall n > N \quad \text{and} \quad \xi_N > \eta_N.$$

3. There are a unique maximal infinite path $\xi_{\max}$ and a unique minimal path $\xi_{\min}$.

Let X_B be the space of all infinite paths of (V, E). It forms a Cantor set naturally. Define the Vershik map

$$\sigma : X_B \to X_B$$

by

$$\sigma(\xi) = \begin{cases} (\eta_1^{\min}, ..., \eta_n^{\min}, \xi_n + 1, ...) & \text{if } \xi_1, ..., \xi_n \in V_{\max},\ \xi_{n+1} \notin V_{\max} \\ \xi_{\min} & \text{if } (\xi_1, \xi_2, ...) = \xi_{\max}, \end{cases}$$

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Theorem (HPS)

Any minimal Cantor system has a Bratteli-Vershik model as described above. Moreover $K_B = K_0(A_y)$, where K_B is the dimension group associated to the Bratteli diagram $B = (V, E)$, and it exhausts all simple dimension group which is not isomorphic $\mathbb{Z}$.

Cantor system with finitely many minimal subsets

Consider a homeomorphism σ of a Cantor set X such that

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Let us call such a system a k-minimal system.

Some C*-algebras associated to a k-simple system

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- $I = \mathrm{C}_0(X \setminus Y) \rtimes_\sigma \mathbb{Z}$, where $Y = \bigcup_{i=1}^k Y_i$.

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Remark

I is an ideal of A, and also an ideal of $A_{y_1,\ldots,y_k}$. One has the following exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow \bigoplus_{i=1}^k \mathrm{C}(Y_i) \rtimes_\sigma \mathbb{Z} \longrightarrow 0.$$

A few more remarks

Theorem (Poon)

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Consider the six-term exact sequence

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The image of Ind is $(\bigoplus_{i=1}^{k} \mathbb{Z})/\mathbb{Z}(1, \ldots, 1)$.

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The image of Ind is $(\bigoplus_{i=1}^{k} \mathbb{Z})/\mathbb{Z}(1, \ldots, 1)$. Hence A is A$\mathbb{T}$ if and only if $k = 1$.

Bratteli-Vershik Model for k-minimal system

Definition

A Kakutani-Rokhlin partition of (X, σ) consists of pairwise disjoint clopen sets

$$\{Z(l, j);\ 1 \leq l \leq L, 1 \leq j \leq J(l)\}$$

for some natural numbers $J(1), ..., J(L)$ such that

1. $\bigcup_{l,j} Z_{l,j} = X$ and
2. $\sigma(Z(l, j)) = Z(l, j + 1)$ for any $1 \leq j < J(l)$.

For a k-simple system, Kakutani-Rohklin partitions always exist. Moreover

Theorem (HPS)

There are Kakutani-Rokhlin partitions of X

$$\mathscr{P}_n = \{Z(n, l, j);\ 1 \leq l \leq L(n), 1 \leq j \leq J(n, l)\}$$

such that

1. *the sequence* $(Z_n := \bigcup_{l=1}^{L(n)} Z(n, l, J(n, l)))$ *is a decreasing sequence of clopen sets with intersection* $\{y_1, y_2, ..., y_k\}$
2. *the partition* $\mathscr{P}_{n+1}$ *is finer than the partition* $\mathscr{P}_n$,
3. $\bigcup_n \mathscr{P}_n$ *generates the topology of* X.

Hence Bratteli-Vershik Model always exists for a k-simple system.

Definition

Let $k \in \mathbb{N}$. The Bratteli diagram B is said to be *k-simple* if for each $n \geq 1$, there are pairwise disjoint subsets $V_1^n, ..., V_k^n$ of V^n such that

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Moreover, denote by $V_o^n = V^n \setminus (V_1^n \cup \cdots \cup V_k^n)$ for $n \geq 1$. Then

1. The diagram B is said to be *strongly k-simple* if for any level n, there is $m > n$ such that if a vertex in V_o^m is connected to some vertex of V_o^n, then it is connected to all vertices of V_o^n.

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2. The diagram B is said to be *non-elementary* if for any V_o^n, there is $m > n$ such that the multiplicity of the edges between V_o^n and V_o^m is either 0 or at least 2.

How to order it?

Definition

An ordered Bratteli diagram $B = (V, E, \geq)$ is called *k-simple* (with a slight abusing of notation) if it satisfies the following conditions:

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2. There are infinite paths $z_{1,\max}, \dots, z_{k,\max}$ and $z_{1,\min}, \dots, z_{k,\min}$ such that for any level n and $1 \leq i \leq k$,

$$\{z^n_{i,\min}, z^n_{i,\max}\} \subset V^n_i$$

and $X_{\max} = \{z_{1,\max}, \dots, z_{k,\max}\}$, $X_{\min} = \{z_{1,\min}, \dots, z_{k,\min}\}$.

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Remark

One consequence of this condition is that there is L such that for all $n \geq L$ and any $v \in V^n_o$, the maximal edge (or minimal edge) starting with v backwards to V^1 will end up in V^1_i for some $1 \leq i \leq k$. Denote by $m_+(v) = i$ (or $m_-(v) = i$).

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3-b if e is an edge with $e \notin E_{\max}$, $r(e) = v$ and $s(e) \in V_i^{n-1}$ with $n \geq 3$, one has

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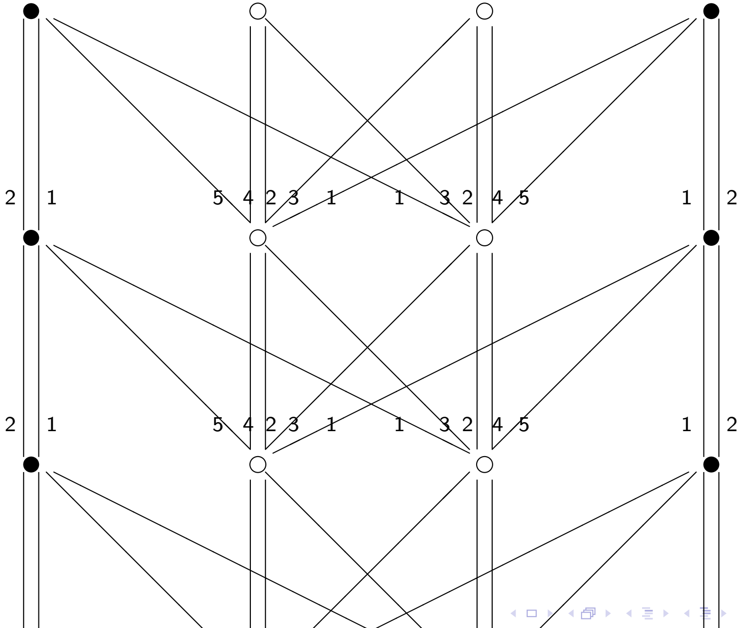
Remark

Note that if $k = 1$, then Condition 3 is redundant.

An example



Theorem

There is a bijection correspondence between the equivalence classes of k-simple ordered Bratteli diagrams and the pointed topological conjugacy classes of Cantor systems with k minimal invariant subsets.

Transition Graphs

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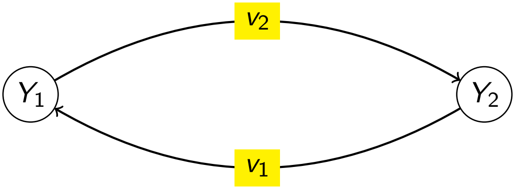
$$m_-(v) = i \quad \text{and} \quad m_+(v) = j.$$

Example

Considering the previous example of 2-simple Bratteli diagram, its transition graph at level n is



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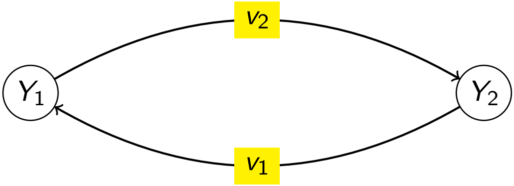

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Lemma

Let $B = (V, E, \geq)$ be k-simple non-elementary ordered Bratteli diagram with $k \geq 2$, and let L_n denotes the transition graph of B at level n. Then, if there is an edge v_1 has the vertex Y_i as the source point, then there is a closed walk $(v_1, ..., v_n(= v_1))$ in L_n.

Index map and transition graph

Recall that the index map

$$\bigoplus_{i=1}^{k} \mathbb{Z} \cong \bigoplus_{i=1}^{k} \mathrm{K}_1(\mathrm{C}(Y_i)) \to \mathrm{K}_0(I)$$

is nonzero if $k \geq 2$. Denote by d_i the the image of i-th copy of $\mathbb{Z}$.

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Let Y_i be a minimal component of (X_B, σ), and let L_n be the transition graph of B at level n. Denote by

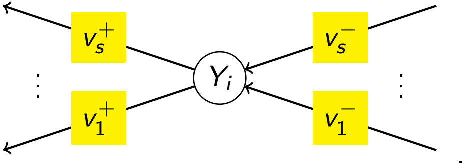
$$E_+(Y_i) = \{v_1^+, ..., v_s^+\}$$

the the set of edges of L_n which have Y_i as source, and denote by

$$E_-(Y_i) = \{v_1^-, ..., v_t^-\}.$$

the the set of edges of L_n which have Y_i as range.

That is,



Theorem

The element d_i is given by

$$(e_{v_1^+} + \cdots + e_{v_s^+}) - (e_{v_1^-} + \cdots + e_{v_t^-}),$$

where e_v stands for $(0, \ldots, 0, 1, 0, \ldots, 0)) \in \bigoplus_{V_o^n} \mathbb{Z}$ with entry 1 at the position v.

Some consequences

Corollary

Let $B = (V, E, \geq)$ be a k-simple ordered Bratteli diagram with $k \geq 2$. Then each transition graph L_n is connected.

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Proof.

The only relation between $d_1, ..., d_k$ is $d_1 + \cdots + d_k = 0$. □

Corollary

Assume B is non-elementary. The transition graph L_n has at least k edges. In particular, one has that

$$|V_o^n| = \left| V_n \setminus \bigcup_{i=1}^{k} V_i^n \right| \geq k$$

for all n.

Corollary

If B is a non-elementary ordered Bratteli diagram, then

$$\mathrm{Image}(\mathrm{Ind}) \cap K_0^+(I_B) = \{0\}.$$

Moreover, if B is assume to be strongly k-simple (so the ideal I_B is simple), then the image of the index map is in subgroup of I_B which consists of infinitesimal elements.

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Corollary

Denote by r the $\mathbb{Q}$-rank of I_B. Then $r \geq k$ and the cone of positive linear maps from I_B to $\mathbb{R}$ has dimension at most $r - k + 1$.

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Corollary

Let (X, σ) be a indecomposible Cantor system with k minimal subsets. Then the C-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is stably finite. Therefore, if $k \geq 2$, it is a stably finite C*-algebra with stable rank 2 and real rank 0.*

Which unordered Bratteli diagram carries such an order?

Consider the k-simple ordered Bratteli diagram $(V, E, \geq)$. The transition graphs $\{L_n;\ n = 2, ...\}$ are compatible to the unordered Bratteli diagram (V, E) in the following sense:

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3. if w (as a vertex in V_o^{n+1}) is connected to some vertex in V_i^n for some $1 \leq i \leq k$, then $(v_1, v_2, ..., v_l)$ passes through Y_i, and
4. for any edge v of L_n, the vertex v (as a vertex in the Bratteli diagram) is connected to some vertex in $V_{\min(v)}^{n-1}$ and is also connected to some vertex in $V_{\max(v)}^{n-1}$.

Theorem

If there is a sequence of directed graphs $\{L_n; n = 2, 3, ...\}$ *such that the vertices of each* L_n *are* $\{Y_1, ..., Y_k\}$*, the edges of each* L_n *are labelled by the vertices in* V_o^n*, and* (L_n) *are compatible with* (V, E) *in the sense above, then there is an order on* (V, E) *so that it is a* k*-simple ordered Bratteli diagram.*

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3. for each $v \in V_o^n$, one has that

$$|\{1 \leq i \leq k;\ d_i(v) \neq 0\}| = 0 \text{ or } 2,$$

and if

$$\{1 \leq i \leq k;\ d_i(v) \neq 0\} = \{i_1, i_2\},$$

then $(d_{i_1}(v), d_{i_2}(v))$ is either $(+1, -1)$ or $(-1, +1)$;

Theorem

Let $B = (V, E)$ be an unordered strongly k-simple Bratteli diagram satisfying the condition that any vertex in V_o^{n+1} is connected to all vertices in V^n.
Suppose that there are element $d_1, ..., d_k \in I_B \subseteq K_0(B)$ satisfying the previous conditions. Then there is an order $\geq$ such that $(V, E, \geq)$ is an ordered (strongly) k-simple Bratteli diagram.