On the classification of non-simple graph C^*-algebras with real rank zero

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Outline

Classification of C^*-algebras

Observation (cf. Jordan-Hölder)

When a C^*-algebra A has finitely many ideals a finite decomposition series

$$0 = I_0 \trianglelefteq I_1 \trianglelefteq \cdots \trianglelefteq I_n = A, \qquad I_j/I_{j-1} \text{ is simple}$$

exists with $(I_1/I_0, I_2/I_1, \ldots, I_n/I_{n-1})$ unique up to isomorphism and permutation.

Problem

Suppose I_j/I_{j-1} are all classifiable by K-theory. Is A classifiable by K-theory?

Task

- Find conditions on K-theory to ensure that

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$$

- Find conditions on K-theory to ensure that

$$A \cong B$$

- Find the range of the invariant

Simple sub-quotients

AF-algebra

Direct limit of finite dimensional C^*-algebras.

Elliott

$$(K_0(A), K_0(A)_+) \cong (K_0(B), K_0(B)_+)$$

if and only if

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K}.$$

Kirchberg algebra

Separable, nuclear, purely infinite simple C^*-algebra satisfying the UCT.

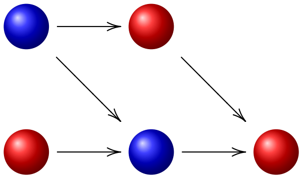
Kirchberg-Phillips

$$(K_0(A), K_1(A)) \cong (K_0(B), K_1(B))$$

if and only if

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K}.$$

Task



Task

- Find conditions on K-theory to ensure that $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$
- Find conditions on K-theory to ensure that $A \cong B$
- Find the range of the invariant

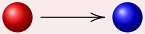
One non-trivial ideal

Elliott, Rørdam, Eilers-Restorff-R

$K^+_{\mathrm{six}}(A; I)$:

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{\iota_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(A/I) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(A/I) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[\iota_*]{} & K_1(I) \end{array}$$

is a complete stable isomorphism invariant provided that $K_0(A)$ has the lexicographic ordering in the case



Filtered K-theory

Let A be a C^*-algebra with finitely many ideals. Suppose we have ideals $I_1 \trianglelefteq I_2 \trianglelefteq I_3$ of A. Then

$$0 \to I_2/I_1 \to I_3/I_1 \to I_3/I_2 \to 0$$

is a short exact sequence of C^*-algebras. Hence, we get

$$\begin{array}{ccccc} K_0(I_2/I_1) & \xrightarrow{\iota_*} & K_0(I_3/I_1) & \xrightarrow{\pi_*} & K_0(I_3/I_2) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(I_3/I_2) & \xleftarrow[\pi_*]{} & K_1(I_3/I_1) & \xleftarrow[\iota_*]{} & K_1(I_2/I_1) \end{array}$$

$K^+_{\text{ideal}}(A)$ is the collection of all K-groups, equipped with order on K_0 and the natural transformations $\{\iota_*, \pi_*, \partial\}$.

Simple

$K^+_{\text{ideal}}(A)$ is the ordered K-theory of A,

$$(K_0(A), K_0(A)_+, K_1(A))$$

One-ideal

$K^+_{\text{ideal}}(A)$ is the six-term exact sequence in K-theory

$$\begin{array}{ccccc} K_0^+(I) & \xrightarrow{\iota_*} & K_0^+(A) & \xrightarrow{\pi_*} & K_0^+(A/I) \\ \uparrow \partial & & & & \downarrow \partial \\ K_1(A/I) & \xleftarrow[\pi_*]{} & K_1(A) & \xleftarrow[\iota_*]{} & K_1(I) \end{array}$$

induced by

$$0 \to I \to A \to A/I \to 0$$

Isomorphism

$$K^{+}_{\text{ideal}}(A) \cong K^{+}_{\text{ideal}}(B)$$

if there exists a lattice isomorphism

$$\beta : \mathrm{Lat}(A) \to \mathrm{Lat}(B)$$

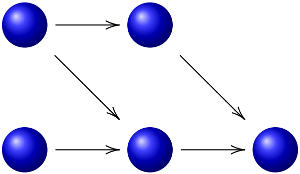
and for all $I_1 \trianglelefteq I_2$ ideals of A, there exists a group isomorphism

$$\alpha_*^{I_1, I_2} : K_*(I_2/I_1) \to K_*(\beta(I_2)/\beta(I_1))$$

preserving all natural transformations and order.

$$\begin{array}{ccccc}
K_0(I_2/I_1) & \xrightarrow{\iota_*} & K_0(I_3/I_1) & \xrightarrow{\pi_*} & K_0(I_3/I_2) \\
\uparrow \partial & & & & \downarrow \partial \\
K_1(I_3/I_2) & \xleftarrow[\pi_*]{} & K_1(I_3/I_1) & \xleftarrow[\iota_*]{} & K_1(I_2/I_1)
\end{array}$$

Purely infinite C^*-algebras



Kirchberg

Let A and B be C^*-algebras with primitive ideal space X. Then

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$$

if and only if

$$A \sim_{\mathrm{KK}_X} B,$$

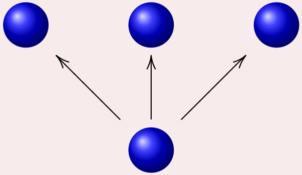
i.e., there exists an invertible element in $\mathrm{KK}(X; A, B)$.

Question

Does $K^+_{\mathrm{ideal}}(A) \cong K^+_{\mathrm{ideal}}(B)$ imply that $A \sim_{\mathrm{KK}_X} B$?

Non-real rank zero counterexample

Primitive ideal space



Counterexample (Meyer-Nest)

There exist A and B such that

- $K^+_{\text{ideal}}(A) \cong K^+_{\text{ideal}}(B)$
- $A \not\sim_{\text{KK}_X} B$.

Consequently, $A \otimes \mathbb{K} \ncong B \otimes \mathbb{K}$.

Arklint-Restorff-R

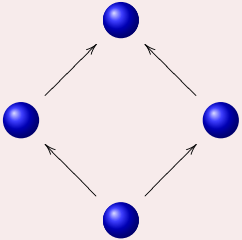
If $RR(A) = RR(B) = 0$, then

$$K^+_{\text{ideal}}(A) \cong K^+_{\text{ideal}}(B) \quad \Longleftrightarrow \quad A \sim_{\text{KK}_X} B.$$

Consequently, $K^+_{\text{ideal}}(A) \cong K^+_{\text{ideal}}(B)$ if and only if $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

Real rank zero counterexample

Primitive ideal space



Counterexample

There exist A and B with real rank zero such that

- $K^+_{\text{ideal}}(A) \cong K^+_{\text{ideal}}(B)$
- $A \not\sim_{\mathrm{KK}_X} B$.

Consequently, $A \otimes \mathbb{K} \ncong B \otimes \mathbb{K}$.

Arklint-Bentmann-Katsura

If $RR(A) = RR(B) = 0$ and the K_1-groups are free, then

$$K^+_{\text{ideal}}(A) \cong K^+_{\text{ideal}}(B) \quad \Longleftrightarrow \quad A \sim_{\mathrm{KK}_X} B.$$

Consequently, $K^+_{\text{ideal}}(A) \cong K^+_{\text{ideal}}(B)$ if and only if $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

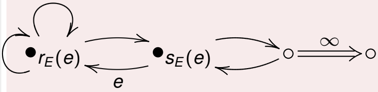
Directed graphs

Definition

A graph $E = (E^0, E^1, r_E, s_E)$ consists of

- a countable set of vertices E^0;
- a countable set of edges E^1; and
- functions $s_E, r_E : E^1 \to E^0$.

Example



Definition

- E^0_{sing}: v is a **singular vertex** if v is a sink or v emits infinitely many edges.
- E^0_{reg}: v is a **regular vertex** if v is not a singular vertex.

Graph C^*-algebras

Graph C^*-algebra

A directed graph $E = (E^0, E^1, r_E, s_E)$ defines a C^*-algebra $C^*(E)$ given as the universal C^*-algebra generated by **projections** $\{p_v \ : \ v \in E^0\}$ and a **partial isometries** $\{s_e \ : \ e \in E^1\}$ satisfying the *Cuntz-Krieger relations:*

(1) $p_v p_w = 0$ for all $v, w \in E^0$ with $v \neq w$;

(2) $s_e^* s_f = 0$ for all $e, f \in E^1$ with $e \neq f$;

(3) $s_e^* s_e = p_{r_E(e)}$ and $s_e s_e^* \leq p_{s_E(e)}$ for all $e \in E^1$;

(4) for every $v \in E^0_{\mathrm{reg}}$,

$$p_v = \sum_{e \in s_E^{-1}(v)} s_e s_e^*$$

Properties of graph C^*-algebras

Theorem (Dichotomy)

For a simple graph C^*-algebra $C^*(E)$

(red ball)	E has no loops	$C^*(E)$ is AF
(blue ball)	All vertices in E can reach at least two loops	$C^*(E)$ purely infinite

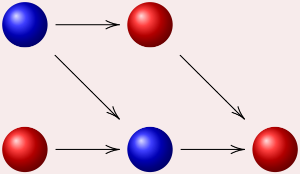
Theorem

If $C^*(E)$ has finitely many ideals, then

- every ideal of I is isomorphic to a graph C^*-algebra
- every quotient of $C^*(E)$ is isomorphic to a graph C^*-algebra.

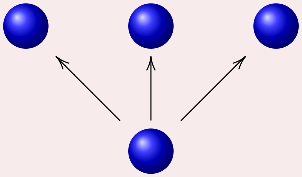
Corollary

$C^*(E)$:



Non-real rank zero counterexample revisited

Primitive ideal space



Counterexample (Meyer-Nest)

There exist A and B such that

- $K^+_{\text{ideal}}(A) \cong K^+_{\text{ideal}}(B)$
- $A \otimes \mathbb{K} \ncong B \otimes \mathbb{K}$.

Question

Can A and B be graph C^*-algebras?

Theorem

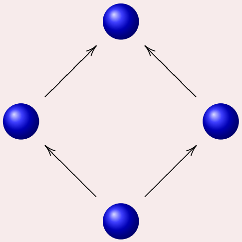
Graph C^*-algebras with finitely many ideals have real rank zero.

Arklint-Restorff-R

A and B are not graph C^*-algebras!

Real rank zero counterexample revisited

Primitive ideal space



Counterexample

There exist A and B with real rank zero such that

- $K^+_{\mathrm{ideal}}(A) \cong K^+_{\mathrm{ideal}}(B)$
- $A \otimes \mathbb{K} \ncong B \otimes \mathbb{K}$.

Question

Can A and B be graph C^*-algebras?

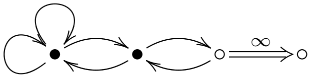
K-theory of graph C^*-algebras

Drinen-Tomforde

$$K_0(C^*(E)) \cong \operatorname{coker}\left(\begin{bmatrix} B^t - \mathrm{id} \\ C^t \end{bmatrix}\right) \quad \text{and} \quad K_1(C^*(E)) \cong \ker\left(\begin{bmatrix} B^t - \mathrm{id} \\ C^t \end{bmatrix}\right)$$

where the adjacency matrix of E with respect to the decomposition $E^0 = E^0_{\mathrm{reg}} \sqcup E^0_{\mathrm{sing}}$ is of the form

$$A_E = \begin{bmatrix} B & C \\ * & * \end{bmatrix}$$



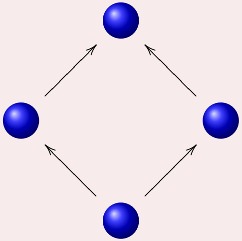
$$A_E = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \infty \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Free K_1

Corollary

The K_1 groups of each quotient of $C^*(E)$ is free.

Primitive ideal space



Counterexample

There exist A and B with real rank zero such that

- $K^+_{\text{ideal}}(A) \cong K^+_{\text{ideal}}(B)$
- $A \otimes \mathbb{K} \ncong B \otimes \mathbb{K}$.

Question

Can A and B be graph C^*-algebras?

Arklint-Bentmann-Katsura

A and B are not graph C^*-algebras!

Conjecture (Eilers-Restorff-R)

If $C^*(E)$ and $C^*(F)$ are graph C^*-algebras with finitely many ideals. Then

$$C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$$

if and only if

$$K^+_{\text{ideal}}(C^*(E)) \cong K^+_{\text{ideal}}(C^*(F))$$

(at least when $K^+_{\text{ideal}}(C^*(F))$ and $K^+_{\text{ideal}}(C^*(F))$ are finitely generated).

One non-trivial ideal

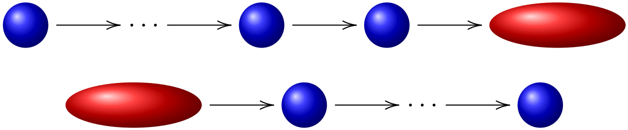
Elliott, Rørdam, Eilers-Tomforde

$K^+_{\mathrm{ideal}}(C^*(E))$:

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{\iota_*} & K_0(C^*(E)) & \xrightarrow{\pi_*} & K_0(C^*(E)/I) \\ \uparrow \partial & & & & \downarrow 0 \\ K_1(C^*(E)/I) & \xleftarrow[\pi_*]{} & K_1(C^*(E)) & \xleftarrow[\iota_*]{} & K_1(I) \end{array}$$

determines $C^*(E)$ up to stable isomorphism among all graph C^*-algebras with a unique non-trivial ideal. The order of $K_0(C^*(E))$ is redundant unless $C^*(E)$ is AF.

Linear ideal lattice



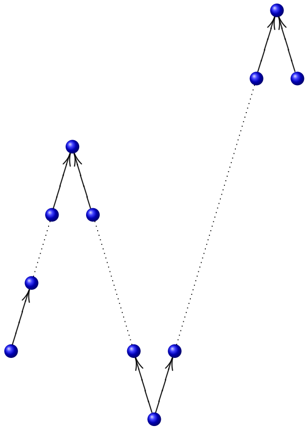
Elliott, Eilers-Restorff-R

$K^+_{\text{ideal}}(C^*(E))$ determines $C^*(E)$ up to stable isomorphism among all graph C^*-algebras of the above form. The order of $K_0(C^*(E))$ is redundant unless $C^*(E)$ is AF.

Accordian spaces

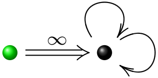
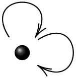
Bentmann-Köhler

$A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$ if and only if $K^{+}_{\text{ideal}}(A) \cong K^{+}_{\text{ideal}}(B)$.



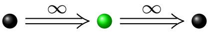
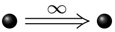
Source Removable

Source Removable (S)

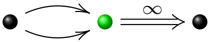
E		
$C^*(E)$	3 ideals	2 ideals

Reduction

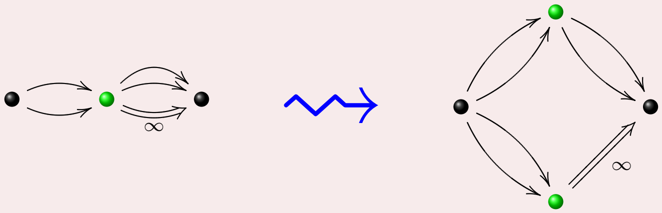
Reduction (R)

E		
$C^*(E)$	4 ideals	3 ideals

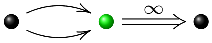
Out-splitting



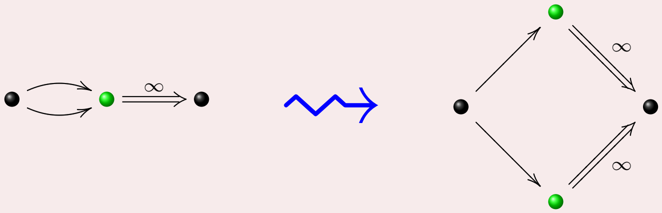
Out-splitting (O)



In-splitting



In-splitting (I)



Moves (S), (R), (O), (I) and Morita equivalence

Theorem

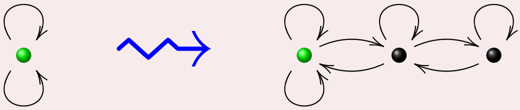
Moves (S) and (R) induce strongly Morita equivalent graph C^*-algebras.

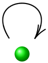
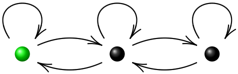
Bates-Pask

Moves (O) and (I) induce strongly Morita equivalent graph C^*-algebras.

Cuntz-Splice

Cuntz-Splice



E		
$C^*(E)$	$C(\mathbb{T})$	Purely infinite

Cuntz-Splice and Morita Equivalence

Cuntz-Rørdam, Restorff

Move (C) induces strongly Morita equivalent Cuntz-Krieger algebras satisfying Condition (K).

Definition

A is a Cuntz-Krieger algebra if $A = C^*(E)$ where E is a finite graph with no sinks and no sources.

Eilers-Sørensen-R, (work in progress)

Move (C) induces strongly Morita equivalent graph C^*-algebras, for unital graph C^*-algebras.

Classification Cuntz-Krieger algebras

Restorff

Let $C^*(E)$ and $C^*(F)$ be Cuntz-Krieger algebras with finitely many ideals. Then

$$C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$$

if and only if

$$K^+_{\text{ideal}}(C^*(E)) \cong K^+_{\text{ideal}}(C^*(F)).$$

Corollary

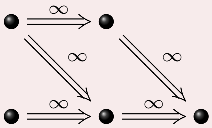
The equivalence relation

$$E \sim_{\text{ME}} F \quad \Longleftrightarrow \quad C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$$

on the class of finite graphs with no sinks and sources satisfying Condition (K) is the equivalence relation generated by Moves (S), (R), (O), (I), and (C).

Classification of Amplified graphs

Amplified graphs



Eilers-Sørensen-R

The equivalence relation

$$E \sim_{\mathrm{ME}} F \quad \Longleftrightarrow \quad C^*(E) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$$

on the class of amplified graphs with finitely many vertices is the equivalence relation generated by Moves (S), (R), (O), and (I).