On generalized Powers-St∅rmer's Inequality

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Plan of talk

1. Background (from Quantam information theory)

2. Formulation

3. Double piling structure of matrix monotone functions and matrix convex functions

4. Chracterizations of the trace property

Background

1. Total error probability:

$$\begin{aligned}
\rho_1, \rho_2 &: \ \textit{hypothetic states on } \mathbf{C}^d \\
&: \text{density matrix on } \mathbf{C}^d, \text{that is} \\
&\rho_i \geq 0, \mathrm{Tr}(\rho_i) = 1 \ (i = 1, 2) \\
E = \{E_1, E_2\} &: \ \textit{quantum multiple test} \\
&: \ d \times d \text{projections } E_1 + E_2 = 1 \\
\mathrm{Succ}_i(E) &:= \mathrm{Tr}(\rho_i E_i) \ (i = 1, 2) \\
\mathrm{Err}_i(E) &:= 1 - \mathrm{Succ}_i(E) = \mathrm{Tr}(\rho_i(1 - E_i)) \\
\mathrm{Err}(E) &:= \frac{1}{2}\mathrm{Tr}(\rho_1 E_2) + \frac{1}{2}\mathrm{Tr}(\rho_2 E_1) \\
&= \frac{1}{2}\{1 - \mathrm{Tr}(E_1(\rho_1 - \rho_2))\}
\end{aligned}$$

2. Assymptotic error exponent for ρ_1 and ρ_2

$$\forall n \in \mathbf{N} \quad E_{(n)} : d^n \times d^n \text{quantum multiple test}$$

$$\mathrm{Err}_n(E_n) := \frac{1}{2}\{1 - \mathrm{Tr}(E_{(n)}(\rho_1^{\otimes n} - \rho_2^{\otimes n}))\}$$

If the limit $\lim_{n\to\infty} -\frac{1}{n}\log \mathrm{Err}_n(E_{(n)})$ exists, we refer to it as *the asymptotic error exponent*.

3. *The quantum Chernoff bound* for ρ_1 and ρ_2

$$\xi_{QCB}(\rho_1, \rho_2) := -\log \inf_{0\leq s\leq 1} \mathrm{Tr}(\rho_1^{1-s}\rho_2^s).$$

Theorem 1. (M. Nussbaum and A. Szkola 2006, K. M. R. Audenaert, et al.2006)

Let $\{\rho_1, \rho_2\}$ be hypothetic states on $\mathbf{C}^d$ and $E_{(n)}$ be a support projections on $(\rho_1^{\otimes n} - \rho_2^{\otimes n})$. Then one has

$$\xi_{QCB} = \lim_{n\to\infty} -\log \mathrm{Err_n}(E_{(n)})$$

In the proof of Theorem 1 the following inequality played a kye role.

Theorem 2. (K. M. R. Audenaert et al. 2011) For any positive matrices A and B on $\mathbf{C}^d$ we have

$$\frac{1}{2}(\mathrm{Tr}A + \mathrm{Tr}B - \mathrm{Tr}|A-B|) \leq \mathrm{Tr}(A^{1-s}B^s)\ (s \in [0,1]).$$

When $s = \dfrac{1}{2}$, Powers and Størmer proved the inequality in 1970.

Formulation

If we consider a function $f(t) = t^{1-s}$ and $g(t) = t^s = \dfrac{t}{f(t)}$, then the previous inequality can be reformed by

(1)
$$\frac{1}{2}(\mathrm{Tr}A+\mathrm{Tr}B-\mathrm{Tr}|A-B|) \leq \mathrm{Tr}(f(A)^{\frac{1-s}{2}}g(B)f(A)^{\frac{1-s}{2}})$$

Problem 3. Let $n \in \mathbf{N}$. When the inequality holds for any $n \times n$ positive definite matrices A and B ?

For $0 \leq s \leq 1$ since the function $t \mapsto t^s$ is operator monotone on $[0, \infty)$, we may hope that the inequality holds when f is operator monotone on $[0, \infty)$.

Definition 4. 1. A function f is sait to be *matrix convex of order* n or n*-convex* in short (resp. *matrix concave of order* n or n*-concave*) whenever the inequality

$$f(\lambda A+(1-\lambda)B) \leq \lambda f(A)+(1-\lambda)f(B),\ \lambda \in [0,1]$$

(resp. $f(\lambda A + (1-\lambda)B) \geq \lambda f(A) + (1-\lambda)f(B),\ \lambda \in [0,1]$) holds for every pair of selfadjoint matrices $A, B \in M_n$ such that all eigenvalues of A and B are contained in I.

2. A function f is said to be *Matrix monotone functions* on I are similarly defined as the inequality

$$A \leq B \Longrightarrow f(A) \leq f(B)$$

for any pair of selfadjoint matrices $A, B \in M_n$ such that $A \leq B$ and all eigenvalues of A and B are contained in I.

We call a function f *operator convex* (resp. *operator concave*) if for each $k \in \mathbb{N}$, f is k-convex (resp. k-concave) and *operator monotone* if for each $k \in \mathbb{N}$ f is k-monotone.

Example 5. Let $f(t) = t^2$ on $(0, \infty)$. It is well-known that f is not 2-monotone. We now show that the function f does not satisfy the inequality (1). Indeed, let us consider the following matrices

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then we have

$$AB^{-1}A = \frac{2}{3}A.$$

Set $\tilde{A} = A \oplus \mathrm{diag}(\underbrace{1, \cdots, 1}_{n-2}), \tilde{B} = B \oplus \mathrm{diag}(\underbrace{1, \cdots, 1}_{n-2})$ in M_n. Then, $\tilde{A} \leq \tilde{B}$ and for any positive linear function φ on M_n

$$
\begin{aligned}
\varphi(f(\tilde{A})^{\frac{1}{2}}g(\tilde{B})f(\tilde{A})^{\frac{1}{2}}) &= \varphi(\tilde{A}\tilde{B}^{-1}\tilde{A}) \\
&= \varphi(\frac{2}{3}A \oplus \mathrm{diag}(\underbrace{1, \cdots, 1}_{n-2})) \\
&< \varphi(A \oplus \mathrm{diag}(\underbrace{1, \cdots, 1}_{n-2})) \\
&= \varphi(\tilde{A}).
\end{aligned}
$$

On the contrary, since $\tilde{A} \leq \tilde{B}$, from the inequality (1) we have

$$
\varphi(\tilde{A}) + \varphi(\tilde{B}) - \varphi(\tilde{B} - \tilde{A}) \leq 2\varphi(f(\tilde{A})^{\frac{1}{2}}g(\tilde{B})f(\tilde{A})^{\frac{1}{2}}),
$$

or

$$
\varphi(\tilde{A}) \leq \varphi(f(\tilde{A})^{\frac{1}{2}}g(\tilde{B})f(\tilde{A})^{\frac{1}{2}}),
$$

and we have a contradiction. □

Theorem 6. (D. T. Hoa-O-H. M. Toan 2012)

Let f be a $2n$-monotone function on $[0, \infty)$ such that $f((0, \infty)) \subset (0, \infty)$. Then for any pair of positive matrices $A, B \in M_n(\mathbf{C})$

$$\mathrm{Tr}(A)+\mathrm{Tr}(B)-\mathrm{Tr}(|A-B|) \leq 2\mathrm{Tr}(f(A)^{\frac{1}{2}}(A)g(B)f(A)^{\frac{1}{2}})$$

The point of the proof is the n-monotonicity of g.

Double piling structure of matrix monotone functions and matrix convex functions

1. $P_n(I)$: the spaces of n-monotone functions
2. $P_\infty(I)$: the space of operator monotone functions
3. $K_n(I)$: the space of n-convex functions
4. $K_\infty(I)$: the space of operator convex functions

The we have

$$P_1(I) \supseteq \cdots \supseteq P_{n-1}(I) \supseteq P_n(I) \supseteq P_{n+1}(I) \supseteq \cdots \supseteq P_\infty(I)$$
$$K_1(I) \supseteq \cdots \supseteq K_{n-1}(I) \supseteq K_n(I) \supseteq K_{n+1}(I) \supseteq \cdots \supseteq K_\infty(I)$$

$$P_{n+1}(I) \not\subseteq P_n(I) \qquad K_{n+1}(I) \not\subseteq K_n(I)$$

$$P_\infty = \cap_{n=1}^{\infty} P_n(I) \qquad K_\infty = \cap_{n=1}^{\infty} K_n(I)$$

Theorem 7. Let consider the following three assertions.

(i) $f(0) \leq 0$ and f is n-convex in $[0, \alpha)$,

(ii) For each matrix a with its spectrum in $[0, \alpha)$ and a contraction c in the matrix algebra M_n,

$$f(c^\star ac) \leq c^\star f(a)c,$$

(iii) The function $\frac{f(t)}{t}$ $(= g(t))$ is n-monotone in $(0, \alpha)$.

1. (Hansen-Pedersen:1985) Three assertions are equivalent if f is operator convex. In this case a function g is operator monotone.

2. (O-Tomiyama:2009)

$$(i)_{n+1} \prec (ii)_n \sim (iii)_n \prec (i)_{[\frac{n}{2}]},$$

where denotion $(A)_m \prec (B)_n$ means that "if (A) holds for the matrix algebra M_m, then (B) holds for the matrix algebra M_n".

Using an idea in [Hansen-Pedersen:1985] we can show the following result.

Proposition 8. (D. T. Hoa-O-H. M. Toan 2012) Under the same condition in Theorem 7 consider the following assetions.

(iv) f is $2n$-monotone.

(v) The function $\frac{t}{f(t)}$ is n-monotone in $(0, \alpha)$.

We have, then, $(iv)_{2n} \prec (v)_n$.

Theorem 6: Let f be a $2n$-monotone function on $[0,\infty)$ such that $f((0,\infty)) \subset (0,\infty)$. Then for any pair of positive matrices $A, B \in M_n(\mathbf{C})$

$$\mathrm{Tr}(A)+\mathrm{Tr}(B)-\mathrm{Tr}(|A-B|) \leq 2\mathrm{Tr}(f(A)^{\frac{1}{2}}(A)g(B)f(A)^{\frac{1}{2}})$$

Sketch of the proof:

A, B : positive matrices
$A - B = (A-B)_+ - (A-B)_- = P - Q$,
$|A - B| = P + Q$.
We may show that

$$\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}}(A)g(B)f(A)^{\frac{1}{2}}) \leq \mathrm{Tr}(P)$$

holds.

$$\begin{aligned}
&\mathrm{Tr}(A) - \mathrm{Tr}(f(A)^{\frac{1}{2}}(A)g(B)f(A)^{\frac{1}{2}}) \\
&= \mathrm{Tr}(f(A)^{\frac{1}{2}}g(A)f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}}(A)g(B)f(A)^{\frac{1}{2}}) \\
&\leq \mathrm{Tr}(f(A)^{\frac{1}{2}}g(B+P)f(A)^{\frac{1}{2}}) - \mathrm{Tr}(f(A)^{\frac{1}{2}}(A)g(B)f(A)^{\frac{1}{2}}) \\
&\leq \mathrm{Tr}(f(B+P)^{\frac{1}{2}}(g(B+P)-g(B))f(B+P)^{\frac{1}{2}}) \\
&\leq \mathrm{Tr}(f(B+P)^{\frac{1}{2}}g(B+P)f(B+P)^{\frac{1}{2}}) - \mathrm{Tr}(f(B)^{\frac{1}{2}}g(B)f(B)^{\frac{1}{2}} \\
&= \mathrm{Tr}(P)
\end{aligned}$$

Since any C*-algebra can be realized as a closed selfadjoint $*$-algebra of $B(H)$ for some Hilbert space H. We can generalize Theorem 6 in the framework of C*-algebras.

Theorem 9. (D. T. Hoa-O-H. M. Toan 2012)

Let τ be a tracial functional on a C^*-algebra $\mathcal{A}$, f be a strictly positive, operator monotone function on $[0, \infty)$. Then for any pair of positive elements $A, B \in \mathcal{A}$

$$\tau(A) + \tau(B) - \tau(|A - B|) \leq 2\tau(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}),$$

where $g(t) = tf(t)^{-1}$.

Chracterizations of the trace property

The generalized Powers-Størmer inequality implies the trace property for a positive linear functional on operator algebras.

Lemma 10. (D. T. Hoa-O-H. M. Toan 2012)

Let φ be a positive linear functional on M_n and f be a continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f((0, \infty)) \subset (0, \infty)$. If the following inequality

(2) $\varphi(A+B) - \varphi(|A-B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}})$

holds true for all $A, B \in M_n^+$, then φ should be a positive scalar multiple of the canonical trace Tr on M_n, where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$.

Let φ be a positive linear functional on M_n and $s \in [0,1]$. From Lemma 10 it is clear that if the following inequality

$$\varphi(A+B) - \varphi(|A-B|) \le 2\varphi(A^{\frac{1-s}{2}} B^s A^{\frac{1-s}{2}}) \tag{3}$$

holds true for any $A, B \in M_n^+$, then φ is a tracial. In particular, when $s = 0$ the following inequality characterizes the trace property

$$\varphi(B) - \varphi(A) \le \varphi(|A-B|) \quad (A, B \in M_n^+). \tag{4}$$

From this observation we have

Corollary 11. (Stolyarov-Tikhonov-Sherstnev:2005) Let φ be a positive linear functional on M_n and the following inequality

$$\varphi(|A+B|) \le \varphi(|A|) + \varphi(|B|) \tag{5}$$

holds true for any self-adjoint matrices $A, B \in M_n$. Then φ is a tracial.

Corollary 12. (Gardner:1979) Let φ be a positive linear functional on M_n and the following inequality

$$|\varphi(A)| \leq \varphi(|A|) \tag{6}$$

holds true for any self-adjoint matrix $A \in M_n$. Then φ is a tracial.

Theorem 13. (D. T. Hoa-O-H. M. Toan 2012)

Let φ be a positive normal linear functional on a von Neumann algebra $\mathcal{M}$ and f be a continuous function on $[0, \infty)$ such that $f(0) = 0$ and $f((0, \infty)) \subset (0, \infty)$. If the following inequality (7)

$$\varphi(A) + \varphi(B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}})$$

holds true for any pair $A, B \in \mathcal{M}^+$, then φ is a trace, where $g(t) = \begin{cases} \frac{t}{f(t)} & (t \in (0, \infty)) \\ 0 & (t = 0) \end{cases}$.

Let $\mathcal{A}$ be a von Neumann algebra and φ be a positive linear functional on $\mathcal{A}$. In the case of the inequality (7) the set $P(\mathcal{A})$ is not enough as a testing set.

Indeed, let p, q be arbitrary orthogonal projections from a von Neumann algebra $\mathcal{M}$. Since $q \geq p \wedge q$ it follows that $pqp \geq p(p \wedge q)p = p \wedge q$. So $pqp \geq p \wedge q$ holds for any pair of projections. From that it follows

$$\varphi(p+q-|p-q|) = 2\varphi(p \wedge q) \leq 2\varphi(pqp) = 2\varphi(f(p)^{\frac{1}{2}}g(q)f(p)^{\frac{1}{2}})$$

Corollary 14. Let φ be a positive linear functional on a C^*-algebra $\mathcal{A}$ and f be a continuous function on $[0,\infty)$ such that $f(0)=0$ and $f((0,\infty))\subset(0,\infty)$. If the following inequality

(8)

$$\varphi(A)+\varphi(B)-\varphi(|A-B|)\leq 2\varphi(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}})$$

holds true for any pair $A,B\in\mathcal{A}^+$, then φ is a tracial functional, where $g(t)=\begin{cases}\frac{t}{f(t)} & (t\in(0,\infty))\\ 0 & (t=0)\end{cases}$.

Take the universal representation π of $\mathcal{A}$ and consider enveloping von Neumann algebra $\mathcal{M}=\pi(\mathcal{A})''$. Apply the previous Theorem to the normal positive functional $\hat{\varphi}$ on $\mathcal{M}$ such that $\hat{\varphi}(\pi(A))=\varphi(A)$ for $A\in\mathcal{A}$.

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