Noncommutative Geometry, Positive Scalar Curvature, and the Strong Novikov Conjecture

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Ancient Mathematics

Geometry (Euclid): Studying the relations between figures or objects in the two dimensional plane (resp. the three dimensional space). This is called plane geometry (resp. solid geometry).

Objects: points, lines, triangles, parallelograms, etc.

Relations: congruent ($\cong$), similar ($\sim$), congruent to a part of ($\leq$), commensurable, etc.

Algebra: How to solve equations with one unknown or several unknowns (including Diophantus equations) of different degrees

Rene Descarte: Ananlytic geometry

Geometry $\longleftrightarrow$ Algebra

point in the plane $\longleftrightarrow$ pair of real numbers (x, y)

point in the space $\longleftrightarrow$ triple of real numbers (x, y, z)

$$\left\{\begin{matrix} \text{a straight line} \\ \text{in the plane} \end{matrix}\right\} \quad \longleftrightarrow \quad \left\{\begin{matrix} \text{a linear equation} \\ \text{of two unknowns} \end{matrix}\right\}$$

$$\left\{\begin{matrix} \text{a straight line} \\ \text{in the space} \end{matrix}\right\} \quad \longleftrightarrow \quad \left\{\begin{matrix} \text{a system of two linear} \\ \text{equations of three unknowns} \end{matrix}\right\}$$

$$\text{a plane in the space} \longleftrightarrow \left\{\begin{matrix} \text{a linear equation} \\ \text{of three unknowns} \end{matrix}\right\}$$

Cartesian Geometry: associate a given space with some

functions

$$X = \text{Euclidean plane} \longleftrightarrow \left\{ \begin{array}{l} \text{two functions} \\ x\colon X{\to}\mathbb{R}, \quad y\colon X{\to}\mathbb{R} \end{array} \right\}$$

$$X = \left\{ \begin{array}{l} \text{Euclidean three} \\ \text{dimensional space} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{three functions} \quad x\colon X{\to}\mathbb{R} \\ \qquad y\colon X{\to}\mathbb{R}, \quad z\colon X{\to}\mathbb{R} \end{array} \right\}$$

Study a space by means of some (canonical) functions

on the space.

Cartesian Geometry is a bridge from constant mathe-

matics to variable mathematics

Riemann is perhaps the first person who had the idea that our universe may not be a Euclidean space of dimension three. In his opinion, our daily experiences only tell us that locally we have freedom of degree three in our space (back and forth, left and right, up and down), or of degree four in our space-time. So he initiated the study of spaces that have the same number of degrees of freedom around any point, ie, that locally look like the Euclidean space $\mathbb{R}^n$. This is the concept of a manifold.

Definition. A topological space M is called a **manifold** if for every point $p \in M$, there is an open neighborhood U_p of p such that U_p is homeomorphic to $\mathbb{R}^n$.
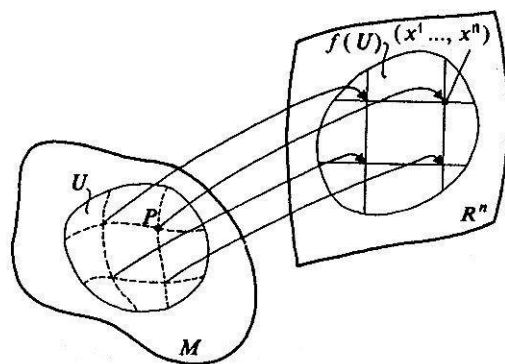

Examples
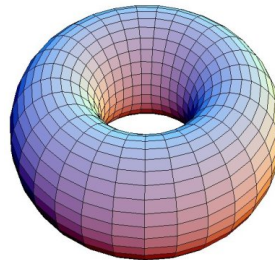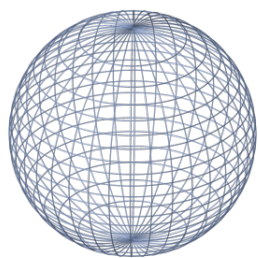
Unlike in Cartesian geometry, which has a single coordinate system, on a manifold we may not be able to choose global coordinates; we only have local coordinates. This means there is no canonical way to choose coordinate functions.

In order to do calculus on a manifold, we need an extra structure called a differential structure.
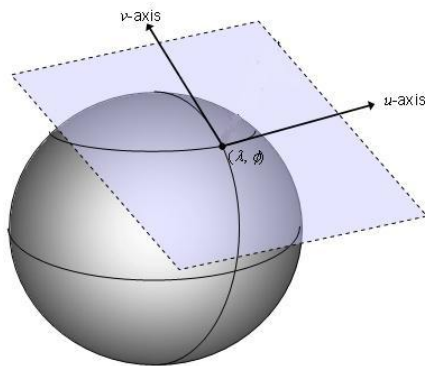
Differential Manifold: The coordinate functions can be (and should be) chosen in a compatible way in the following sense
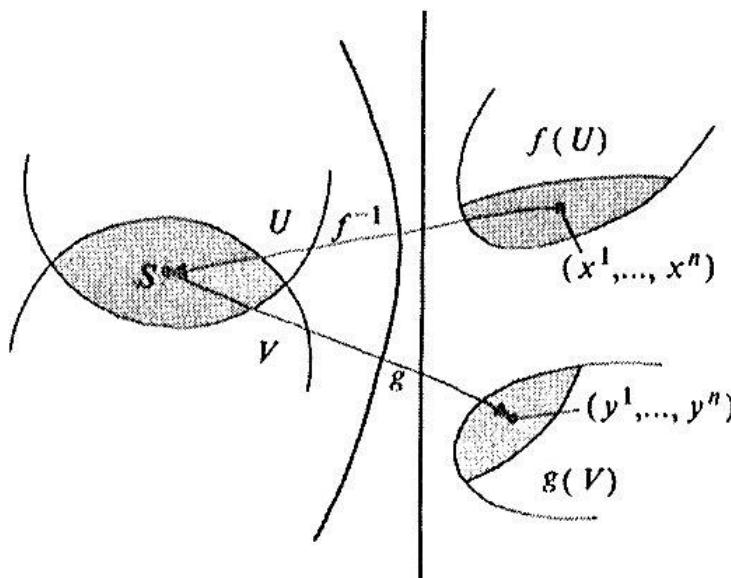
where $g \circ f^{-1}$ is differentiable.

If $f : U \to \mathbb{R}^n$, and $g : V \to \mathbb{R}^n$ are local coordinates for U and V, then $\quad g \circ f^{-1} : f(U \cap V) \longrightarrow g(U \cap V)$ is differentiable as a function from an open set of $\mathbb{R}^n$ to another open set of $\mathbb{R}^n$.

Tangent space

Associate to each point $p \in M$ an n-dimensional real vector space $T_p M = \mathbb{R}^n$.

Definition. Let $C^\infty(M)$ be the collection of all smooth functions $f : M \to \mathbb{R}$. A **tangent vector** V at point $p \in M$ is a derivation $V : C^\infty(M) \to \mathbb{R}$ satisfying

$$V(fg) = f(p)V(g) + V(f)g(p) \qquad \forall f, g \in C^\infty(M).$$

(Use the algebra $C^\infty(M)$ to define a geometric object)

At any point t_0 of a smooth curve $\alpha(t)$, there is a tangent vector $\alpha'(t_0) \in T_{\alpha(t_0)}M$, defined by

$$(\alpha'(t_0))(f) = \frac{d}{dt} f(\alpha(t))|_{t_0} \quad \forall f \in C^\infty(M)$$

8

A <u>Riemannian manifold</u> (M, g) is a differential manifold M on which each tangent space is equipped with an inner product g such that g varies smoothly from point to point.

The length of the curve $\{\alpha(t); \ a \leq t \leq b\}$ is defined to be $\int_a^b < \alpha'(t), \alpha'(t) >^{\frac{1}{2}} dt$.

Define a metric d on M by

$$d(p, q) = \inf\{\ \text{length}(\alpha) \mid \alpha \text{ connects } p \text{ and } q\} \qquad \forall p, q \in M$$

Scalar Curvature: Let M be an n-dimensional Riemannian manifold. For $p \in M$, let $B_r(M, p)$ be an open ball with radius r and center p. Then, there is a number $k(p)$ satisfying the following condition:

$$\frac{\text{volume}(B_r(M, p))}{\text{volume}(B_r(\mathbb{R}^n, 0))} = 1 - \frac{k(p)}{6(n+2)}r^2 + o(r^2) \ ,$$

where $\mathbb{R}^n$ is given the standard Euclidean metric. Such $k(p)$ is called the **scalar curvature** of M at p.
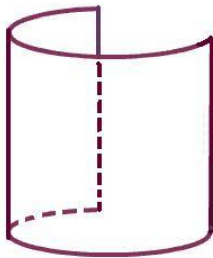
$k(p) > 0$: locally smaller than Euclidean space.

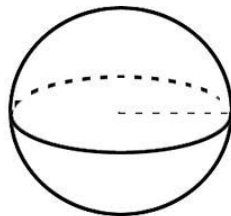$k(p) < 0$: locally larger than Euclidean space.

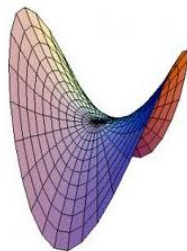For dimension 2:

scalar curvature $= 2 \times$ Gauss Curvature.

- a cylinder has curvature 0

- a sphere has positive curvature

- a saddle surface has negative curvature

Invariants: $\pi_1(X), \quad H_n(X), \quad H^n(X)$.

Numerical invariants: Euler number $\chi(M)$, Signature $signature(M)$.

$$\chi(M) \triangleq \dim(H^0(M) \otimes \mathbb{R}) - \dim(H^1(M) \otimes \mathbb{R}) + \dim(H^2(M) \otimes \mathbb{R}) - \cdots$$

Traditionally: Euler number of a polyhedron is defined to be $\#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces})$

$$\chi(S^2) = 2 \qquad\qquad \chi(T^2) = 0$$

<u>Gauss-Bonnet Theorem.</u> Let M be a closed oriented 2-dimensional Riemannian manifold. Let $k(p)$ be the Gauss curvature of M. Then

$$\int_M k(p) = 2\pi\chi(M)$$

For compact orientable surface

$M \neq S^2 \Leftrightarrow$ the universal covering space $\widetilde{M}$ of M is contractible

$\Leftrightarrow$ Euler $(M) \leq 0$

$\Leftrightarrow \nexists$ metric with everywhere positive Gauss curvature

Definition. A manifold M with fundamental group

$\pi_1(M) = \Gamma$ is called a $K(\Gamma, 1)$ manifold if $\pi_n(M) = 0$ for

all $n \geq 2$ (or equivalently, the universal covering space

$\widetilde{M}$ (of M) is contractible).

Theorem If a 2-dimensional closed manifold M is a

$K(\pi_1(M), 1)$ manifold, then M can not have everywhere

positive Gauss curvature.

For higher dimensional manifolds, replace Gauss cur-

vature with scalar curvature.

Gromov-Lawson Conjecture: Any compact $K(\Gamma, 1)$ manifold M does not have everywhere positive scalar curvature.

This conjecture is a special case of **Gromov Positive Scalar Curvature Conjecture** which involves non compact complete Riemannian manifolds (called open manifolds).

Even to prove Gromov-Lawson Conjecture, we need to use the universal covering $\tilde{M}$ (of M) which is an open manifold.

Let M be an orientable compact manifold of dimension 4k. Then $H^{4k}(M, \mathbb{R}) = \mathbb{R}$.

$$\cup : \ H^{2k}(M, \mathbb{R}) \times H^{2k}(M, \mathbb{R}) \to H^{4k}(M, \mathbb{R}) = \mathbb{R}.$$

defines a quadratic form $<,>$ on $H^{2k}(M, \mathbb{R})$.

Write $\quad < x, y >= \displaystyle\sum_{i=1}^{t} x_i y_i - \sum_{j=t+1}^{t+m} x_j y_j \ .$

Define signature of M to be the signature of the quadratic form $t - m$.

Like the Euler number, the signature is a homotopy invariant.

There is a characteristic class called the Hirzebruch class $\mathcal{L}(M) \in \oplus_{t=1}^{k} H^{4t}(M, \mathbb{R})$, such that

$$signature(M) = \int_M \mathcal{L}(M) = <\mathcal{L}(M), [M] > \in \mathbb{Z}.$$

Higher signature:

Let M be an n-dimensional manifold with $\pi_1(M) = \Gamma$.

Let $B\Gamma$ be the classifying space of Γ. The identification of $\pi_1(M)$ with Γ gives a classifying map $u : M \to B\Gamma$.

For each $x \in H^*(B\Gamma, \mathbb{R})$, one can define higher Signature

$$signature_x(M, u) = <\mathcal{L}(M) \cup u^*x, \ [M] >$$

$$= \int_M \mathcal{L}(M) \cup u^*x \ \in \ \mathbb{R}$$

Novikov Conjecture. Higher Signature is a homotopy invariant for manifolds M. That is, for any discrete group Γ and any $x \in H^*(B\Gamma, \mathbb{R})$, for any two n-dimensional manifolds M, N with $\pi_1(M) = \Gamma$, $\pi_1(N) = \Gamma$, and classifying map $u : M \to B\Gamma$, if there is a homotopy equivalence $f : N \to M$, then

$$signature_x(M, u) = signature_x(N, u \circ f).$$

The Novikov conjecture is known to hold for many discrete groups.

One of the most powerful methods to prove both Gromov-Lawson conjecture and Novikov Conjecture is to use C^*-algebras as non commutative spaces.

Index theory and noncommutative geometry.

Many numerical invariants of manifolds can be written as indices of elliptic operators.

Let $E_i \to M (i = 1, 2)$ be a smooth vector bundles over M, and let $C^\infty(M, E_i)$ be the smooth sections of E_i. An Elliptic Operator on M is a differential operator $D : C^\infty(M, E_1) \to C^\infty(M, E_2)$ with invertible leading symbol—or roughly speaking, it is invertible modulo lower order operators.

$$D \text{ is elliptic } \implies D \text{ is Fredholm}$$

That is, $ker(D)$ and $coker(D)$ are finite dimensional.

Define $Index(D) = \dim ker(D) - \dim coker(D)$.

(D is invertible modulo compact operators, i.e., $\exists T$ such that $DT - I$ and $TD - I$ are compact operators.)

For example:

$$index(\text{deRham operator}) = \chi(M)$$

$$index(\text{Signature operator}) = signature(M)$$

Atiyah—Singer index theory implies

Index(Dirac Operator)=$\hat{A}(M) - -\hat{A}$ genus of M.

Theorem. (Lichnerowicz formula) For the Dirac operator D on a smooth manifold M

$$D^2 = \nabla^*\nabla + \frac{1}{4}k,$$

where k is the scalar curvature function on M.

Consequently, if compact manifold M has positive scalar curvature then Dirac operator D is invertible.

Corollary. (Atiyah-Singer) For a compact manifold M, if $\hat{A}(M) \neq 0$, then M can not have positive scalar curvature everywhere.

But an elliptic operator on an open manifold is no longer Fredholm, i.e., it is not invertible modulo compact operators, so one can no longer define its index.

Atiyah-Brown-Douglas-Fillmore-Kasparov work out a way to define K-homology group $K_*(M)$ in terms of "abstract elliptic operators". In this language an elliptic opetator D on M defines a K-cycle $[D] \in K_0(M)$ and

$$Index: \ [D] \in K_0(M) \longrightarrow K_0(\{pt\}) = \mathbb{Z}$$

here $K_0(\{pt\})$ could be regarded as K-theory $K_0(\mathcal{K}(H))$ of the algebra $\mathcal{K}(H)$ of all compact operators on an infinite dimensional Hilbert space H.

The index map is the map $\pi_* : K_0(M) \to K_0(\{pt\})$, where $\pi : M \to \{pt\}$ is the quotient map identifying the whole manifold M.

In order for a continuous map $f : X \to Y$ to induce a map $f_* : K_*(X) \to K_*(Y)$, one needs f to be proper, that is, the pre-image of any compact set needs to be compact. So if M is an open manifold, then $\pi_* : K_0(M) \to K_0(\{pt\})$ does not make sense for $\pi : M \to \{pt\}$, as we said elliptic operators on a non-compact manifold may not be Fredholm.

So we need to use another kind of quotient space: we need to identify any bounded set to a single point, but not the whole manifold.

Example. $\mathbb{Z} \cong \mathbb{R} \ncong \{pt\}$.

$\mathbb{Z} = \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

$\mathbb{R} = \text{——————————————}$

Using the ideas of Descartes and Riemann, one can study a space X by studying the functions on X.

$$X \text{ a compact space} \longrightarrow C(X) = \left\{ \begin{array}{c} \text{continuous complex} \\ \text{valued functions} \end{array} \right\}$$

$$= \text{commutative } C^*\text{-algebras.}$$

$$M \text{ a differential manifold} \longrightarrow C^\infty(M) \subset C(M)$$

$$= \left\{ \begin{array}{c} \text{commutative } C^*\text{-algebra} \\ \text{with certain dense subalgebra} \end{array} \right\}.$$

We can regard general noncommutative C^*-algebras as noncommutative topological spaces.

Noncommutative C^*-algebras with certain dense subalgebras (called smooth subalgebras) can be regarded as noncommutative differential manifolds.

Let us use Connes' noncommutative quotient to explain the idea.

Noncommutative quotient spaces (Connes):

$X = \{x, y\}, \quad \sim$ identifying x and y

$X/\sim = \{pt\}, \ C(X/\sim) = \mathbb{C}$

Noncommutative quotient:

$C(X) = \mathbb{C} \oplus \mathbb{C} \longrightarrow M_2(\mathbb{C})$ by

$$(a, b) \longmapsto \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$x \sim y$: means $p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \sim q := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

The partial isometry $v := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ has initial space p and final space q.

Put v (and v^*) in, then $p(= v^*v) \sim q(= vv^*)$.

$M_2(\mathbb{C})$—the Noncommutative quotient space

$$K_0(M_2(\mathbb{C})) = K_0(\mathbb{C}) = K_0(\mathcal{K})$$

M :　complete Riemannian manifold

Noncommutative quotient space $C^*(M)$ (Roe algebra):

Closure of the set of all <u>locally compact</u> bounded linear

operators $T \in B(L^2(M))$ (or $B(L^2(M, E))$, where E is

the spinor bundle on which D acts) with <u>finite propagation</u>.

Locally compact: If $f \in C_0(M)$, then TM_f and $M_f T$

are compact, where $M_f \in B(L^2(M, E))$ is defined by

$M_f g = f \cdot g$, for $g \in L^2(M, E)$.

Finite propagation: There is a $r > 0$ such that if

$\mathrm{dist}(supp(f), supp(g)) > r$, then $M_f T M_g = 0$,　or

$$supp(Ts) \subset \{x \in M;\ \mathrm{dist}(x, supp(s)) \le r\},$$

$$\forall s \in L^2(M)(\text{ or }\ L^2(M, E)).$$

One can define Roe algebra for general metric space.
Then we have $C^*(\mathbb{R}) = C^*(\mathbb{Z})$.

Even though an elliptic operator D on an open manifold $\tilde{M}$ is not invertible modulo compact operators, it is invertible modulo $C^*(\tilde{M})$. One can define

$$Index: \quad K_*(\tilde{M}) \longrightarrow K_*(C^*(\tilde{M})).$$

In general, one can define

$$Index: KX_*(Y) \longrightarrow K_*(C^*(Y)),$$

where $KX_*(Y)$ is coarse K-homology of space Y.

Coarse Geometric Novikov (CGN) Conjecture The above
index map is injective up to tensoring with $\mathbb{Q}$.

CGN Conjecture for a group Γ (or equivalently, for the universal cover $\tilde{M}$ of $K(\Gamma, 1)$ manifold M)$\Rightarrow$ Gromov-Lawson conjecture for $K(\Gamma, 1)$ manifolds.

Let Γ be a discrete group, and $\mathbb{C}\Gamma$ be the group algebra of Γ with complex coefficients. One can define the C^*-algebra $C^*_{max}\Gamma$ to be the completion of $\mathbb{C}\Gamma$ with respect to a certain maximum norm $\|\cdot\|_{max}$.

Let $E\Gamma$ be the classifying space of Γ for free actions. One can define an index map, called Baum-Connes map:

$$\mu : K^{\Gamma}_*(E\Gamma) \to K_*(C^*_{max}\Gamma).$$

Strong Novikov conjecture: μ is rationally injective.

Strong Novikov Conjecture for $\Gamma \Rightarrow$ Both Novikov conjecture for manifolds with π_1 to be Γ and Gromov-Lawson Conjecture for $K(\Gamma, 1)$ manifolds.

Let Γ be a residually finite group with

$$\Gamma \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots$$

with Γ/Γ_i finite and $\cap_{i=1}^{\infty}\Gamma_i = \{1\}$, where Γ_i are normal subgroups.

Define the box space $X(\Gamma) = \cup_{i=1}^{\infty}\Gamma/\Gamma_i$ with

$$\lim_{\substack{n+m\to\infty \\ n\neq m}} d(\Gamma/\Gamma_n, \Gamma/\Gamma_m) = \infty.$$

<u>Theorem</u> (G-Wang-Yu) If $E\Gamma/\Gamma$ has the homotopy type of a finite CW complex, then the strong Novikov conjecture for Γ and all Γ_n, with $n = 1, 2 \cdots \Longleftrightarrow$ Coarse geometric Novikov conjecture for $X(\Gamma)$.

This gives a geometrization of the strong Novikov conjecture for these groups.

Yu proved the coarse geometric Novikov conjecture for spaces that can be uniformly embedded into a Hilbert Space

Dranishnikov-G-Lafforgue-Yu constructed a discrete metric space which can not be uniformly embedded into a Hilbert space, answering an open question of Gromov negatively.

Gromov proved that any discrete metric space containing a sequence of expanders can not be uniformly embedded into Hilbert space.

Kasparov-Yu proved the coarse geometric Novikov conjecture for spaces that can be uniformly embedded into a uniformly convex Banach space.

Lafforgue constructed residually finite groups Γ with property T, such that the box spaces $X(\Gamma)$ can not be uniformly embedded into uniformly convex Banach spaces. Each of the spaces $X(\Gamma)$ contains a sequence of expanders.

Corollary. (G-Wang-Yu) For Lafforgue's examples $X(\Gamma)$, the coarse geometric Novikov conjecture holds.

This is the first result to show that certain spaces containing sequences of expanders satisfy the coarse geometric Novikov conjecture.