Noncommutative Geometry and Kadison-Singer Algebras

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What is *Noncommutative Geometry ?*

Geometrical:

Classification of group actions on a manifold M/G,
e.g., $\mathbb{Z}$ acts on S^1 by rotations: $n : e^{2\pi it} \to e^{2\pi i(t+n\theta)}$; to classify $S^1/\theta\mathbb{Z}$.

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Algebraic:

Geometrical and topological invariants of the algebra $C(M) \times G$, $C^\infty(M) \times G$ or $L^\infty(M) \times G$,
e.g., dimension, K-theory, (co)homology groups, etc.

Classical geometry:

$S^1 = \mathbb{R}/\mathbb{Z}$ and $\mathbb{R} = \tilde{S}^1$

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$$\begin{array}{ccc} \mathbb{Z} = \hat{S}^1 & \to & \mathbb{C}[\mathbb{Z}] \\ & & \Updownarrow \\ S^1 = \hat{\mathbb{Z}} & \to & \mathscr{P}(S^1) \\ & & \text{alg. geo.} \end{array}$$

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$$\begin{array}{llll} \mathbb{Z} = \hat{S}^1 & \to \mathbb{C}[\mathbb{Z}] & \to C^*(\mathbb{Z}) \\ & \quad\Updownarrow & \quad\Updownarrow \\ S^1 = \hat{\mathbb{Z}} & \to \mathscr{P}(S^1) & \to C(S^1) \\ & \text{alg. geo.} & C^*\text{-alg} \end{array}$$

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\begin{array}{cccccccc}
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S^1 = \hat{\mathbb{Z}} & \to & \mathcal{P}(S^1) & \to & C(S^1) & \to & L^\infty(S^1) \\
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\text{geometry} & \text{alg. geo.} & C^*\text{-alg} & \text{vN alg} & \text{analysis}
\end{array}
$$

Basic facts:
S^1 = maximal ideal space of $C(S^1)$; $C(S^1) =< U = e^{2\pi it} : U^*U = 1 >$;
$C(S^1 \times S^1) =< U, V : U^*U = V^*V = 1, UV = VU >$

Definition

Suppose G is a group (discrete or not) and π a unitary representation of G on a Hilbert space $\mathcal{H}$ (e.g., $l^2(\mathbb{Z})$). Then $\mathrm{span}\{\pi(G)\}^-$ is called a *C^*-algebra*; the commutant of $\pi(G)$ (or linear span of all intertwiners) is called a *von Neumann algebra*.

Theorem (Gelfand-Naimark, 1943)

If $\mathfrak{A}$ is an abelian C*-algebra, then $\mathfrak{A} \cong C(\hat{\mathfrak{A}})$ where $\hat{\mathfrak{A}}$ is the maximal ideal space.

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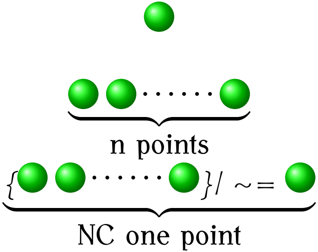
Theorem (von Neumann, 1929)

If $\mathfrak{A}$ is an abelian von Neumann algebra, then $\mathfrak{A} \cong L^\infty([0,1], \mu)$.

noncommutative operator algebras=noncommutative topology or probability theory

Examples

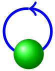


Classical one point: $\longleftrightarrow$ $\mathbb{C}$

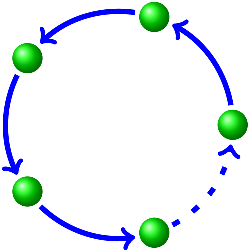
Classical n points: $\longleftrightarrow$ $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$

NC one point $\longleftrightarrow$ $M_n(\mathbb{C})$

NC points



selfmap $\longleftrightarrow$ $C(S^1)$



$\longleftrightarrow$ $M_n(C(S^1))$

$$K_0(M_n(C(S^1)) = \mathbb{Z}, \quad K_1(M_n(C(S^1)) = \mathbb{Z}.$$

Noncommutative torus $S^1 \times_\theta S^1 (= S^1 \times_\theta \mathbb{Z})$

Algebra:

$\mathfrak{A} = C^* \left\langle U, V : V^{-1}UV = e^{2\pi i\theta}U \right\rangle$

θ is rational	$\mathfrak{A} \cong M_n(C(S^1 \times S^1))$, $K_0(\mathfrak{A}) = \mathbb{Z}$
θ is irrational	$K_0(\mathfrak{A}) = \mathbb{Z} + \theta\mathbb{Z}$ (not connected),
	$K_1(\mathfrak{A}) = \mathbb{Z} + \mathbb{Z}$

$\mathbb{A}_{\mathbb{Q}}/C_{\mathbb{Q}}$

Adele ring : $\mathbb{A}_{\mathbb{Q}} = \prod_{p\in\mathscr{P}} \mathbb{Q}_p$ $\quad$ dx: Haar measure on $\mathbb{A}_{\mathbb{Q}}$
Idele class group: $C_{\mathbb{Q}} = \mathbb{A}^*_{\mathbb{Q}}/\mathbb{Q}^*$ $\quad$ d^*x: Haar measure on $C_{\mathbb{Q}}$

$$dx = \lim_{\epsilon\to 0} \epsilon|x|^{1+\epsilon} d^*x$$

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For $h \in \mathcal{S}(C_{\mathbb{Q}})$ and $f \in \mathcal{S}(\mathbb{A})$, define

$$(U(h)f)(x) = \int_{C_{\mathbb{Q}}} f(g^{-1}x)h(g)d^*g.$$

Let $R_\lambda = \hat{\chi}_{[-\lambda,\lambda]}\chi_{[-\lambda,\lambda]}$.

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Conjecture

$$Trace(R_\lambda U(h)) = 2h(1)\log'\lambda + \sum_{p\in\mathcal{P}} \int_{Q_p} \frac{h(g^{-1})}{|1-g|_p} d^*g + o(1).$$

Central Questions

Basic Questions: classification and representation.

Noncommutative euclidean spaces

$\mathbb{C}\langle x_1, \ldots, x_n\rangle$, $x_1, \ldots, x_n$ are non commuting variables; or $\mathbb{C}[F_n]$

C* (or topological) level	von Neumann(measure space) level
$C^*(F_n)$	$\mathcal{L}_{F_n}$
$K_0(C^*(F_n)) = \mathbb{Z}$	$K_0(\mathcal{L}_{F_n}) = \mathbb{R} \quad (n > 1)$
$K_1(C^*(F_n)) = \mathbb{Z}^n$	$K_1(\mathcal{L}_{F_n}) = \{0\}$

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Classification

Is "n" an invariant of the algebra? Can $\mathcal{L}_{F_n}$ be generated by fewer than n elements?

Approximate embedding problem

Can any algebra be approximated by finite dimensional matrix algebra (in terms of a measurement)? i.e., Suppose $\mathfrak{A}$ is an algebra with a linear functional ρ (or a trace). Suppose $\mathfrak{A}$ is generated by $X_1, \ldots, X_d$. For any $\epsilon > 0$ and $N > 0$, is there a large matrix algebra $M_k(\mathbb{C})$ with functional ρ, $A_1, \ldots, A_d$ in $M_k(\mathbb{C})$ such that

$$|\rho(X_{i_1} \cdots X_{i_s}) - \rho(A_{i_1} \cdots A_{i_s})| < \epsilon, \quad \forall s \leq N?$$

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Connes's Embedding: ρ is a trace, $\mathfrak{A}$ is a (separable) factor of type II_1.

Some Known Results

1) Jones Index:
$H \leq G$ a subgroup:

$$H \leq G \qquad\qquad [G:H] \in \mathbb{N} \cup \{\infty\}$$

$vN(G)$ the commutant of the left regular representation of G, $\mathfrak{M} \subset vN(G)$ is a von Neumann subalgebra (weakly closed subalgebra):

$$\mathfrak{M} \subset vN(G) \qquad\qquad [vN(G):\mathfrak{M}] \in \{4\cos^2 \tfrac{\pi}{n} : n \in \mathbb{N}\} \cup [4,\infty]$$

2) Voiculescu's free dimension:
With $(\mathfrak{A}, \rho)$ given, $X_1, \dots, X_d \in \mathfrak{A}$, define

$$\Gamma(X_1, \dots, X_d; \epsilon, k, N) = \{(A_1, \dots, A_d)\} \subset M_k(\mathbb{C})^d \cong \mathbb{R}^{4dk^2}.$$

$$fdim(X_1, \dots, X_d) = \liminf_{\epsilon, k, N} \frac{1}{k^2} \log \frac{vol(\Gamma(X_1, \dots, X_d; \epsilon, k, N)}{vol(ball(\rho(I))} + d.$$

Theorem 1 (Voiculescu)

fdim is an algebraic invariant, i.e.,

$$fdim(X_1, \ldots, X_d) = fdim(Y_1, \ldots, Y_c)$$

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Theorem 2

$fdim(M_n(\mathbb{C})) = 1 - \frac{1}{n^2}$;
$fdim(SL_2(\mathbb{Z})) = \frac{7}{6}$;
$fdim(F_n) = n$;
$fdim(\mathcal{L}_{G \times H}) = 1$, or $-\infty$, when G and H are infinite;
($\mathcal{L}_{F_n}$ are prime factors)
$fdim(\mathcal{L}_{SL_n(\mathbb{Z})}) = 1$, $n \geq 3$. (Shen-Ge)

Other viewpoints

All above is "real" noncommutative geometry.

What is a complex noncommutative geometry?

Motivations

- When we replace $\mathbb{Z}$ by $\mathbb{N}$, the above $l^2(\mathbb{Z}) \cong L^2(S^1)$ becomes $l^2(\mathbb{N}) \cong H^2(\mathbb{D})$, the Hardy space.

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- Algebraic geometry: $\mathcal{I}$ is an ideal in $\mathbb{C}[x_1, \ldots, x_n]$, $Var(\mathcal{I}) = \{x : p(x) = 0, \forall p \in \mathcal{I}\}$. One can replace $\mathbb{C}[x_1, \ldots, x_n]$ by any noncommutative ring.

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- The need to study $\mathbb{R}/\mathbb{N}^*$. For $f \in \mathcal{S}(0, \infty)$,

$$\zeta(s) = \frac{\int_0^\infty \sum_n f(nx) x^{s-1} dx}{\int_0^\infty f(x) x^{s-1} dx}.$$

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- Kadison and Singer (1960): $\mathcal{T}$ is a triangular algebra if $\mathcal{T} \cap \mathcal{T}^*$ (the diagonal of $\mathcal{T}$) is abelian—a generalization of $H^\infty(\mathbb{D})$.

Natural generalization: $\mathfrak{A}\langle X_1, \ldots, X_d\rangle$, $\mathfrak{A}$ a noncommutative coefficient ring; $\mathcal{G}$ a “base” space given by $\mathfrak{A}$; $\phi_i \in \mathfrak{A}\langle X_1, \ldots, X_d\rangle$ are noncommutative polynomials. Define

$$\mathrm{Var}(\{\phi_i\}_i) = \{(P_1, \ldots, P_d) \in \mathcal{G}^d : \phi_i(P_1, \ldots, P_d) = 0\}.$$

Von Neumann's (continuous) geometry:
Points: projections in $\mathcal{B}(\mathcal{H})$.
Manifold: all projections in a von Neumann algebra.

Suppose $\mathfrak{A}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$. Then

$$\mathrm{Lat}(\mathfrak{A}) = \{P \in \mathcal{B}(\mathcal{H}) : (I - P)AP = 0, A \in \mathfrak{A}\}$$

is called a (von Neumann) manifold.
In commutative geometry, $(1 - x)x = 0$ when $x^2 = 0$. Thus $(I - P)AP$ are zero polynomials.
In von Neumann's continuous geometry, a von Neumann algebra $\mathfrak{A}$ is a coefficient ring, $\mathcal{G} = \{\text{all projections in } \mathfrak{A}\}$ (Grassmann manifold) is the base (point) space for $\mathfrak{A}$.
Von Neumann's "manifolds" are lattices of projections in a von Neumann algebra.

—they are "extremely" disconnected since both P and $I - P$ are in a manifold.

Generalizing this idea, one may consider any $\mathfrak{A} \subset M_n(\mathbb{C})$, $\mathcal{G} = \mathcal{G}(\mathfrak{A})$ (ϕ_i are all degree zero polynomials) and define

$$\begin{aligned}\mathrm{Lat}(\mathfrak{A}) &= \{P \in \mathcal{G} \mid A : P \to P, \forall A \in \mathfrak{A}\} \\ &= \{P \mid (1 - P)AP = 0, \forall A \in \mathfrak{A}\}.\end{aligned}$$

$\mathrm{Lat}(\mathfrak{A})$ is always a lattice: $P \wedge Q, P \vee Q \in \mathrm{Lat}(\mathfrak{A})$, $\forall P, Q \in \mathrm{Lat}(\mathfrak{A})$.

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If $\mathcal{V} \subset \mathcal{G}$ is a sublattice or any subset, then we define

$$\begin{aligned}\mathrm{Alg}(\mathcal{V}) &= \{A \in \mathfrak{A} \mid A : P \to P, \forall P \in \mathcal{V}\} \\ &= \{A \mid (1-P)AP = 0, \forall P \in \mathcal{V}\}.\end{aligned}$$

$\mathrm{Alg}(\mathcal{V})$ is always an algebra.

Definition

Suppose $\mathfrak{A} = \mathscr{B}(\mathscr{H}) = M_\infty(\mathbb{C})$ and $\mathscr{P}$ a set of projections in $\mathscr{B}(\mathscr{H})$. We call $\mathrm{Alg}(\mathscr{P})$ a Kadison-Singer algebra and $\mathrm{LatAlg}(\mathscr{P})$ a Kadison-Singer lattice (or KS-manifold) if $\mathrm{LatAlg}(\mathscr{P})$ is a "minimal" generating reflexive lattice for the von Neumann algebra generated by $\mathscr{P}$.

In this case, we denote $\mathrm{Alg}(\mathscr{P})$ by $\mathrm{Naf}(\mathscr{P})$ and $\mathrm{Lat}(\cdot)$ by $\mathrm{Var}(\cdot)$. Then $\mathrm{Alg}(\mathscr{P})$ is a maximal reflexive algebra with respect to its diagonal subalgebra.

Conjecture: If $P \in \mathrm{Var}(\mathfrak{A})$ and $P \neq 0, I$, then $I - P \notin \mathrm{Var}(\mathfrak{A})$.

If there is a minimal KS-manifold containing $\mathscr{P} \subset \mathscr{G}$, then we call $\mathrm{Var}(\mathscr{P})$ the "Zariski closure" of $\mathscr{P}$.

Examples: $\mathfrak{A} = M_n(\mathbb{C})$.

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- If $\mathscr{P} = \{P, Q, R\}$ is a lattice in $\mathfrak{A}$, then is $\mathrm{VarNaf}(\mathscr{P}) = \mathscr{P} \cup \{0, I\}$?—Answer: no!
  This is the simplest non trivial case. The lattice is called a double triangle lattice:

$$\begin{array}{ccccc} & & 1 & & \\ & \nearrow & \uparrow & \nwarrow & \\ P & & Q & & R \\ & \nwarrow & \uparrow & \nearrow & \\ & & 0 & & \end{array}$$

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- If $\mathscr{P} = \{P_1, \ldots, P_n\}$, then what is $\mathrm{VarNaf}(\mathscr{P})$?

The closure of three points

1) Finite-dimensional case

Let $\mathcal{G}(r, n)$ be the Grassmann manifold consisting all r-dimensional subspaces of $\mathbb{C}^n$, which can be identified with all rank r projections in $M_n(\mathbb{C})$.

Suppose P, Q, R are three elements in $\mathcal{G} = \cup_r \mathcal{G}(r, n)$ such that they generate a double triangle lattice (as above).

Theorem (Yuan-Ge)

The KS-manifold generated by a double triangle lattice is homeomorphic to S^2. Zariski closure of any three points in S^2 is S^2.

When n is even, randomly picked three projections in $M_n(\mathbb{C})$ form a double triangle lattice with probability one.

2) The limit case as $n \to \infty$
As $n \to \infty$, P, Q, R converges in distribution, i.e.,

$$\int_{\mathcal{G}^3} \frac{1}{n} \text{trace}(\phi(P, Q, R)) dPdQdR$$

has a limit for any polynomial ϕ.
Because P, Q, R are non commuting variables, they can be modeled by the following elements:

$G_3 = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$: the free product of $\mathbb{Z}_2$ with itself 3 times (or n times, in general).

Let $\mathcal{L}_{G_3}$ be the group von Neumann algebra acting on $l^2(G_3)$.

If U_1, U_2, U_3 are canonical generators for G_3 (or $\mathcal{L}_{G_3}$), then $P_j = \frac{1-U_j}{2}$, $j = 1, 2, 3$, are projections.
$\mathcal{F}_3$: the lattice consisting of P_1, P_2, P_3 and $0, 1$.

Theorem (Yuan-Ge)

$\mathrm{Var}(\mathrm{Naf}(\mathcal{F}_3)) \setminus \{0, 1\}$ is homeomorphic to S^2. Zariski closure of any three elements in S^2 generate S^2.

Theorem (Yuan)

The automorphism group of $\mathscr{L}_{G_3}$ that preserve S^2 is isomorphic to S_3.

Theorem (Hou-Yuan)

The KS-manifold generated by a double triangle lattice in any von Neumann algebra with a trace is homeomorphic to S^2. The only connected KS-manifold in $M_n(\mathbb{C})$ is homeomorphic to S^2.
(to appear in Math. Ann.)

Questions: 1. How does the geometry of S^2 determine $\mathfrak{A}$?
(S^2 "minimally" generates the coefficient ring $\mathfrak{A}$.)
2. What is VarNaf($\mathcal{F}_4$)? Is it finite-dimensional?
3. Are there (nontrivial) abelian KS-algebras?

Product KS-manifolds

Suppose $\mathscr{P}, \mathbb{Q}$ are KS-manifolds. There is a natural way to associate $\mathscr{P} \times \mathbb{Q}$ with KS-algebra $\mathrm{Naf}(\mathscr{P}) * \mathrm{Naf}(\mathbb{Q})$—the free product.
IT SHOULD BE RIGHT
But even with the simplest case $\mathscr{P} = \mathbb{Q} = \{0, P, 1\}$, we do not have a proof.

Thanks!