Classification of extensions of $A\mathbb{T}$ -algebras

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Let A and B be C^* -algebras. Recall that an extension of A by B is a short exact sequence

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The extension (E, α, β) is called trivial, if the above sequence splits, i.e. if there is a homomorphism $\gamma: A \to E$ such that $\beta \circ \gamma = id_A$.

We call (E, α, β) essential, if $\alpha(B)$ is an essential ideal in E. We denote the set of all essential extensions by $\mathcal{E}\mathrm{xt}^e(A, B)$.



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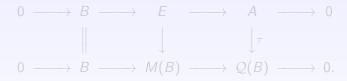
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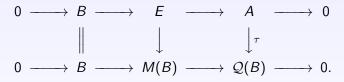


The Busby invariant of (E, α, β) is a homomorphism τ from A into the corona algebra Q(B) = M(B)/B defined by $\tau(a) = \pi(\sigma(b))$ for $a \in A$, where $\pi: M(B) \to \mathcal{Q}(B)$ is the quotient map, and $b \in E$ such that $\beta(b) = a$.



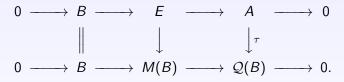
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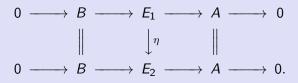
If A is unital and the Busby invariant is unital, then (E, α, β) is called unital.



Let $e_i: 0 \to B \to E_i \to A \to 0$ be two extensions with Busby invariants τ_i for i = 1, 2.

Definition 1

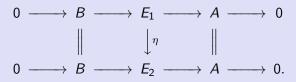
 e_1 and e_2 are called congruent, denoted by $e_1 \equiv e_2$, if there exists an isomorphism η making the following diagram commute:



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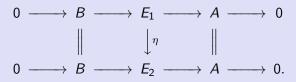
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Definition 2

 e_1 and e_2 are called (strongly) unitarily equivalent, denoted by $e_1 \stackrel{\sim}{\sim} e_2$, if there exists a unitary $u \in M(B)$ such that $\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for all $a \in A$. Denote by $\operatorname{Ext}(A,B)$ or $\operatorname{Ext}_s(A,B)$ the set of (strongly) unitary equivalence classes of extensions of A by B.

Definition 3

Weakly unitarily equivalent, denoted by $e_1 \stackrel{w}{\sim} e_2$, if there exists a unitary $u \in \mathcal{Q}(B)$ such that $\tau_2(a) = u\tau_1(a)u^*$ for all $a \in A$. Denote by $\mathsf{Ext}_w(A,B)$ the set of weakly unitary equivalence classes of extensions of A by B.

Definition 4

 e_1 and e_2 are called isomorphic, denoted by $e_1 \cong e_2$, if there exist isomorphisms β, η, α making the following diagram commute:

$$0 \longrightarrow B \longrightarrow E_1 \longrightarrow A \longrightarrow 0$$

$$\downarrow^{\beta} \qquad \downarrow^{\eta} \qquad \downarrow^{\alpha}$$

$$0 \longrightarrow B \longrightarrow E_2 \longrightarrow A \longrightarrow 0.$$

Denote the morphism of extensions by $(\beta, \eta, \alpha) : e_1 \to e_2$. Denote by $\operatorname{Ext}_I(A, B)$ the set of equivalence classes of extensions up to isomorphism.

Sum of extensions

- $\operatorname{Ext}_{s}(A,B)$ and $\operatorname{Ext}_{w}(A,B)$ are semigroups
- Trivial extensions construct subsemigroups of $\operatorname{Ext}_s(A,B)$ and

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Suppose that B is a stable C^* -algebra. Then the sum of two extensions τ_1 and τ_2 is defined to be the homomorphism $\tau_1 \oplus \tau_2$, where $\tau_1 \oplus \tau_2 : A \to \mathcal{Q}(B) \oplus \mathcal{Q}(B) \subseteq M_2(\mathcal{Q}(B)) \cong \mathcal{Q}(B)$.

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The stable Ext-group Ext(A, B) is the quotient of $Ext_s(A, B)$ by the subsemigroup of trivial extensions. The equivalence class of an extension τ in Ext(A, B) is denoted by $[\tau]$.

If $[au_1] = [au_2]$ in Ext(A,B), then au_1 and au_2 are called stably unitarily equivalent, denoted by $au_1 \overset{ss}{\sim} au_2$.

- $[\tau_1] = [\tau_2]$ iff there are trivial extensions σ_i such that $\tau_1 \oplus \sigma_1 \stackrel{s}{\sim} \tau_2 \oplus \sigma_2$
- If A is a separable nuclear C^* -algebra and B is a σ -unital C^* -algebra, then Ext(A,B) is an abelian group
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- $\equiv \implies \stackrel{s}{\sim} \implies \stackrel{w}{\sim} \implies \stackrel{ss}{\sim}$. Conversely, they do not hold.
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When A is non-unital, let $\mathcal{T}(A)$ be the set of lower-semicontinuous densely defined traces on A equipped with the weakest topology such that the functional $\tau \to \tau(a)$ is continuous for any $a \in A^+$ dominated by a projection. Let $Inv(A) = (K_0(A), K_0(A)^+, \Sigma(A), K_1(A), \mathcal{T}(A), r_A)$, where $\Sigma(A) = \{[p] : p \in P(A)\}\$ is the scale and P(A) is the set of projections in Α.

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Let A and B be two unital simple separable amenable C^* -algebras with stable rank one. We write $EII(A) \cong EII(B)$ if

$$(K_0(A), K_0(A)^+, [1_A], K_1(A), T(A), r_A) \cong (K_0(B), K_0(B)^+, [1_B], K_1(B), T(B))$$

$$T(B) \xrightarrow{\gamma} T(A)$$

$$\downarrow r_A \qquad \qquad \downarrow r_B$$

$$(K_0(B)) \xrightarrow{\alpha_0^*} S(K_0(A))$$

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that is, if there are an isomorphism $\alpha_1: K_1(A) \to K_1(B)$, an order isomorphism $\alpha_0: K_0(A) \to K_0(B)$ such that $\alpha_0([1_A]) = [1_B]$ and an affine homeomorphism $\gamma: T(B) \to T(A)$ such that

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Let $e: 0 \to B \to E \to A \to 0$ be an extension of A by B. Denote by K(e) the six term exact sequence of e in K-theory:

$$K_0(B) \longrightarrow K_0(E) \longrightarrow K_0(A)$$
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If α_* , β_* and λ_* are isomorphisms, then $K(e_1)$ and $K(e_2)$ are called isomorphic, written $K(e_1) \cong K(e_2)$. If $A_1 = A_2 = A$, $B_1 = B_2 = B$ and there is an isomorphism $(id_{K_*(A)}, id_{K_*(B)}, \lambda_*) : K(e_1) \to K(e_2)$, then they are called congruent, written $K(e_1) \equiv K(e_2)$.

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$$K_0(x)(K_0(A)_+\setminus\{0\})\subset K_0(B)_+\setminus\{0\}.$$

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Classification — nonunital case

Some Results

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- Eilers-Restorff-Ruiz, 2009: Suppose that A, B are in a certain class of and B has CFP, then $K_*^+(A) + K(e)$ is a complete invariant for full

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Definition

Let B be separable stable C^* -algebra. Then B is said to have the Corona Factorization Property (CFP) if every full projection in M(B) is M-v equivalent to $1_{M(B)}$.

If B has CFP, then

- Note: every nonunital full extension is absorbing, and every unital full extension is unital-absorbing.
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Lemma (Ortega-Perera-Rordam)

Let B be a separable, unital C^* -algebra with finite decomposition rank. Then $B\otimes \mathcal{K}$ has the corona factorization property.

Corollary

Let B be a unital AT-algebra, then $B \otimes \mathcal{K}$ has CFP.

Lemma (Ortega-Perera-Rordam)

Let B be a separable, unital C^* -algebra with finite decomposition rank. Then $B\otimes \mathcal{K}$ has the corona factorization property.

Corollary

Let B be a unital $A\mathbb{T}$ -algebra, then $B\otimes \mathcal{K}$ has CFP.

Theorem

Let A be a simple $A\mathbb{T}$ -algebra with unit. Suppose that $a \in KK_e(A, A)^{++}$ and $\gamma : T(A) \to T(A)$ is an affine homeomorphism such that

$$K_*(a): (K_0(A), K_0(A)^+, [1_A], K_1(A)) \to (K_0(A), K_0(A)^+, [1_A], K_1(A))$$

is an isomorphism and γ is compatible with $K_0(a)$. It follows that there is an automorphism $\phi:A\to A$ such that $KK(\phi)=a$ in KK(A,A) and $\phi_T=\gamma$.

Lemma (Rordam)

Let A and B be separable nuclear C^* -algebras in $\mathcal N$ with B stable, and let $x_1,\ x_2\in E\mathrm{xt}(A,B)$. Then $K(x_1)=K(x_2)$ in $\mathrm{Hext}(A,B)$ if and only if there exist elements a in KK(A,A) and b in KK(B,B) with $K_*(a)=K_*(id_A)$ and $K_*(b)=K_*(id_B)$ such that $x_1b=ax_2$.

Lemma

Let A and B be simple $A\mathbb{T}$ -algebras with A unital and B stable. Assume that $a \in KK(A,A), \ b \in KK(B,B)$ such that $K_*(a) = id_{K_*(A)}$ and $K_*(b) = id_{K_*(B)}$. Then there are isomorphisms $\alpha : A \to A, \ \beta : B \to B$ such that $KK(\alpha) = a$ and $KK(\beta) = b$.

Theorem

Let A_i and B_i be simple $A\mathbb{T}$ -algebras with A unital and B stable. Suppose that $e_i: 0 \to B_i \to E_i \to A_i \to 0$ are non-unital full extensions. Then the following are equivalent:

- (1) E_1 is isomorphic to E_2 .
- (2) There is an extension isomorphism $(\beta, \eta, \alpha) : e_1 \to e_2$, i.e. $e_1 \cong e_2$.
- (3) The six term exact sequences associated to e_1 and e_2 are isomorphic, i.e. there are isomorphisms $\beta_{\sharp}: Inv(B_1) \to Inv(B_2), \ \eta_*: K_*(E_1) \to K_*(E_2)$ and $\alpha_{\sharp}: Ell(A_1) \to Ell(A_2)$ such that $(\beta_*, \eta_*, \alpha_*): K(e_1) \to K(e_2)$ is an isomorphism.

Theorem

Suppose that A_i are simple $A\mathbb{T}$ -algebras with units, and B_i are stabilizations of unital AF-algebras. Let $e_i: 0 \to B_i \to E_i \to A_i \to 0$ be non-unital full extensions. Then the following are equivalent:

- (1) $E_1 \cong E_2$.
- (2) $e_1 \cong e_2$.
- (3) The six term exact sequences associated to e_1 and e_2 are isomorphic, i.e. there is an isomorphism $(\beta_*, \eta_*, \alpha_*) : K(e_1) \to K(e_2)$ for some isomorphisms $\beta_* : (K_0(B_1), K_0(B_1)^+) \to (K_0(B_2), K_0(B_2)^+)$ and $\alpha_{\sharp} : Ell(A_1) \to Ell(A_2)$.

When is an extension an $A\mathbb{T}$ -algebra?

Given an extension $0 \rightarrow B \rightarrow E \rightarrow A \rightarrow 0$

Question: Let A, B be in a class A of C^* -algebras. Which condition will make E be in A?

$$A = \{AF-algebras\} \Longrightarrow E \in A.$$

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Brown-Effros-Elliott, 1980s

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Lin-Rordam, 1992

Let A and B be $A\mathbb{T}$ -algebras with real rank zero and let e be an extension

- $0 \to B \to E \to A \to 0.$ Then the following three conditions are equivalent:
- (1) E is an $A\mathbb{T}$ -algebra of real rank zero.
- (2) E has real rank zero and stable rank one.
- (3) The index maps $\delta_i: K_i(A) \to K_{1-i}(B)$, i = 0, 1 are both trivial.

Dadarlat-Loring, 1993

Assume that A,B are AD-algebras with real rank zero, $K_1(B)=0$ or $K_1(A)$ torsion free. TFAE:

- (1) E is an AD-algebra of real rank zero.
- (2) RR(E) = 0, st(E) = 1.
- (3) $\delta_i = 0$

Theorem

Suppose that A is an $A\mathbb{T}$ -algebra and B is the stabilization of a unital $A\mathbb{T}$ -algebra. Let $e:0\to B\to E\to A\to 0$ be a non-unital full extension of A by B. Then the following are equivalent.

- (1) E is an $A\mathbb{T}$ -algebra.
- (2) The index maps of e are zero.
- (3) The extension e is quasidiagonal.

Proof.

 $(2) \iff (3)$ and $(1) \implies (3)$ are immediate.

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Lemma 1

Suppose that A and B are $A\mathbb{T}$ -algebras with B stable. Then there is an absorbing trivial extension which is also quasidiagonal.

Proof:

 $(2) \Longleftrightarrow (3)$ and $(1) \Longrightarrow (3)$ are immediate.

We only need to show that $(3) \Longrightarrow (1)$.

Lemma 1

Suppose that A and B are $A\mathbb{T}$ -algebras with B stable. Then there is an absorbing trivial extension which is also quasidiagonal.

Lemma 2

Suppose that A and B are $A\mathbb{T}$ -algebras with B stable. Let $e:0\to B\to E\xrightarrow{\psi} A\to 0$ be an essential trivial extension of A by B. If e is quasidiagonal, then E is an $A\mathbb{T}$ -algebra.

Suppose that e is a quasidiagonal extension. Let $A=\lim_{n\to\infty}(A_n,\iota_n)$, where A_n is isomorphic to a quotient of a circle algebra and ι_n are the inclusion maps. Set $\tau_n=\tau\circ\iota_n$ and $E_n=\pi^{-1}(\tau_n(A_n))$, where τ is the Busby invariant associated to e. Then we have an essential extension e_n of A_n by B

$$0 \rightarrow B \rightarrow E_n \rightarrow A_n \rightarrow 0$$

for every $n \in \mathbb{N}$. Hence, there is a commutative diagram

Since
$$A = \lim_{n \to \infty} (A_n, \iota_n)$$
, it follows that $\tau(A) = \overline{\bigcup_{n=1}^{\infty} \tau_n(A_n)}$.

Therefore, it follows that

$$E = \overline{\bigcup_{n=1}^{\infty} E_n} = \lim_{n \to \infty} E_n.$$

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For each A_n , there is an increasing sequence $\{A_{n,k}\}$ of C^* -subalgebras of A_n such that each $A_{n,k}$ is isomorphic to a finite direct sum of C^* -algebras of the form $M_m(C(X))$ and $\bigcup_{k=1}^{\infty} A_{n,k}$ is dense in A_n , where X is a connected compact subset of the unit circle. Set $\tau_{n,k} = \tau \circ \iota_{n,k}$ and $E_{n,k} = \pi^{-1}(\tau_{n,k}(A_{n,k}))$, where $\iota_{n,k} : A_{n,k} \to A$ is the inclusion map. Let $e_{n,k}$ be the essential extension of $A_{n,k}$ by B:

$$0 \to B \to E_{n,k} \to A_{n,k} \to 0.$$

Obviously, there is a commutative diagram

As the above proof, we have

$$E_n = \overline{\bigcup_{k=1}^{\infty} E_{n,k}} = \lim_{k \to \infty} E_{n,k}.$$

Since e is non-unital and full, then $e_{n,k}$ is a non-unital full extension. Hence $e_{n,k}$ is absorbing. By the above proof, the index maps $\delta_i: K_i(A) \to K_{1-i}(B)$ of e are trivial. Since $\tau_{n,k} = \tau \circ \iota_{n,k}$, then the index maps of $e_{n,k}$ are also trivial. From Lemma 1, it follows that A is quasidiagonal relative to B, so the subalgebra $A_{n,k}$ is also quasidiagonal relative to B. Note that $K_*(A_{n,k})$ is free. Hence, $e_{n,k}$ is a trivial and quasidiagonal extension. It follows from Lemma 2 that $E_{n,k}$ is an $A\mathbb{T}$ -algebra. Therefore, E_n is an $A\mathbb{T}$ -algebra. Consequently, E is an $A\mathbb{T}$ -algebra.

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Classification — unital case

Lemma

Suppose that A and B are C^* -algebras with A unital and B stable. Let $e_i: 0 \to B \stackrel{l_i}{\to} E_i \to A \to 0$ be essential unital extensions. Suppose $\tau_2 = \operatorname{Ad} u \circ \tau_1$ for some unitary u in $\mathcal{Q}(B)$. Let v be a partial isometry in M(B) such that $\pi(v) = u$, and let $p = v^*v$ and $q = vv^*$. Then

$$(K(e_1),[1]_0) \equiv (K(e_2),[q]_0 + [1-p]_0).$$

Let $0 \to B \to E \to A \to 0$ be an extension with index maps δ_0 and δ_1 in its K-theory. We set $G' = \{f([1]_0) | f \in \operatorname{Hom}(\operatorname{Ker}\delta_0, \operatorname{Coker}\delta_1)\}$ and let $\pi : K_0(B) \to \operatorname{Coker}\delta_1$ be the quotient map.

Lemma

Let e_i be essential unital extensions with Busby invariant τ_i . If e_1 is weakly unitarily equivalent to e_2 by a unitary $u \in \mathcal{Q}(B)$. Then

$$(K(e_1),[1]_0)\equiv (K(e_2),[1]_0)$$

if and only if $\pi([u]_1)$ is in G'.



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Lemma

Suppose e_i are essential unital extensions with Busby invariant τ_i and e_1 is weakly unitarily equivalent to e_2 . If the index maps of e_i are trivial and

$$(K(e_1),[1]_0)\equiv (K(e_2),[1]_0),$$

then $[e_1] = [e_2]$ in $\operatorname{Ext}_s^u(A, B)$.

Theorem

Let A_i and B_i be simple $A\mathbb{T}$ -algebras with A_i unital and B_i stable.

Suppose that $e_i: 0 \to B_i \to E_i \to A_i \to 0$ are unital quasidiagonal extensions. Then the following are equivalent:

- $(1) E_1 \cong E_2.$
- (2) There is an extension isomorphism $(\beta, \eta, \alpha) : e_1 \rightarrow e_2$.
- (3) There are isomorphisms $\beta_{\sharp}: Ell(B_1) \to Ell(B_2)$,
- $\eta_*: (K_*(E_1), [1]_0) \to (K_*(E_2), [1]_0)$ and $\alpha_\sharp : \textit{Ell}(A_1) \to \textit{Ell}(A_2)$ such that $(\beta_*, \eta_*, \alpha_*) : (K(e_1), [1]_0) \to (K(e_2), [1]_0)$ is an isomorphism.

Theorem

Let A_i and B_i be simple $A\mathbb{T}$ -algebras with A unital and B stable. Suppose that $e_i: 0 \to B_i \to E_i \to A_i \to 0$ are unital essential extensions. Then the following are equivalent:

- (1) $E_1 \otimes \mathcal{K}$ is isomorphic to $E_2 \otimes \mathcal{K}$.
- (2) There is an extension isomorphism $(\beta, \eta, \alpha) : Se_1 \rightarrow Se_2$.
- (3) The six term exact sequences associated to e_1 and e_2 are isomorphic, i.e. there are isomorphisms $\beta_{\dagger}: Ell(B_1 \otimes \mathcal{K}) \to Ell(B_2 \otimes \mathcal{K})$,
- $\eta_*: \mathcal{K}_*(E_1 \otimes \mathcal{K}) \to \mathcal{K}_*(E_2 \otimes \mathcal{K})$ and $\alpha_\sharp : Ell(A_1 \otimes \mathcal{K}) \to Ell(A_2 \otimes \mathcal{K})$ such that $(\beta_*, \eta_*, \alpha_*) : \mathcal{K}(Se_1) \to \mathcal{K}(Se_2)$ is an isomorphism.

Question

Suppose that $e: 0 \to B \to E \to A \to 0$ is essential extension of $A\mathbb{T}$ -algebras.

- Without the condition "absorption or fullness", e is trivial \implies e is QD or E is an $A\mathbb{T}$ -algebra?
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Thanks