Strongly self-absorbing property for inclusions of C^*-algebras with a finite Watatani index

Hiroyuki Osaka (Ritsumeikan University, Japan)

Joint work with Tamotsu Teruya

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Motivation

In Elliott program to classify nuclear C*-algebras by K-theory data the systematic use of strongly self-absorbing C*-algebras plays a central role. In the purely infinite case the Cuntz algebra $\mathcal{O}_\infty$ is a cornerstone of the Kirchberg- Phillips classification of simple purely infinite C*-algebras [17] [25]. In the stably finite case the Jiang-Su algebra $\mathcal{Z}$ plays a role similar to that of $\mathcal{O}_\infty$. In fact Jiang-Su proved in [12] that simple, infinite dimensional AF algebras and Kirchberg algebras (simple, nuclear, purely infinite and satisfying the Universal Coefficient Theorem) are $\mathcal{Z}$-stable, that is, for any such an algebra A one has an isomorphism $\alpha\colon A \to A \otimes \mathcal{Z}$. Gong, Jiang, and Su proved in [5] that $(K_0(A), K_0(A)^+)$ is isomorphic to $(K_0(A \otimes \mathcal{Z}), K_0(A \otimes \mathcal{Z})_+)$ if and only if $K_0(A)$ is weakly unperforated as an ordered group, when A is a simple C*-algebra. Hence A and $A \otimes \mathcal{Z}$ have isomorphic Elliott invariant if A is simple with weakly unperforated K_0-group, that is, $A \cong A \otimes \mathcal{Z}$ whenever A is classifiable. On the contrary, Rørdam and Toms in [31] and [33] presented examples which have the same Elliott invariant as, but are not isomorphic to, and not $\mathcal{Z}$-absorbing. So it appears plausible that the

Elliott conjecture, which is formulated in [30], holds for all simple, unital, nuclear, separable $\mathcal{Z}$-absorbing C*-algebras.

In this talk we reconsider the $\mathcal{D}$-absorbing property for crossed product of a C*-algebra A with $\mathcal{D}$-absorbing by a finite group action with the Rokhlin property in the framework of inclusion of unital C*-algebras $P \subset A$ of Watatani index finite ([36]) and show that if a faithfule conditional expectation E from A to P has the Rokhlin property in the sense of Kodaka-Osaka-Teruya [18], then P is $\mathcal{D}$-absorbing.

Stronly self-absorbing property

Definition 1. A separable, unital C*-algebra D is called *strongly self-absorbing* if it is infinite-dimensional and the map $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ given by $d \mapsto d \otimes 1$ is approximately unitarily equivalent to an isomorphism $\varphi \colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, that is, there is a suquence $(v_n)_{n\in\mathbf{N}}$ of unitaries in D satisfying

$$\|v_n^*(\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}}(d))v_n - \varphi(d)\| \to 0 \ (n \to \infty) \ \forall d \in \mathcal{D}.$$

A C*-algebra A is called *$\mathcal{D}$-absorbing* if $A \otimes \mathcal{D} \cong A$.

Example 2. 1. (Jiang-Su '99) The Jiang-Su algebra $\mathcal{Z}$ is a direct limits of prime dimension drop algebras $I_{p,q} = \{f \in C([0,1], M_{pq}) \mid f(0) \in 1_p \otimes M_q, f(1) \in M_p \otimes 1_q\}$ for relative prime integers $p, q \geq 2$. Then $\mathcal{Z}$ is strongly self-absorbing.

2. (Toms-Winter '07) UHF algebras of infinite type (for example, an universal UHF algebra $\mathcal{U}_\infty = \Pi_p M_{p^\infty}$), Cuntz algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$, $B \otimes \mathcal{O}_\infty$ (with B UHF of infinite type) are strongly self-absorbing property.

Question 3. Let $P \subset A$ be an inclusion of unital C*-algebras and $E\colon A \to P$ be a conditional expectation of index finite type. That is, there is a quasi-basis $\{(w_i, w_i^*)\}_{i=1}^n \subset A \times A$ such that $x = \sum_{i=1}^n E(xw_i)w_i^* = \sum_{i=1}^n w_i E(w_i^* x)$ for any $x \in A$.

(1) If A is strongly self-absorbing, when P is strongly self-absorbing ?

(2) Let $\mathcal{D}$ is strongly self-absorbing and A $\mathcal{D}$-absorbing. When P is $\mathcal{D}$-absorbing ?

In this talk we introduce the **finitely saturated property** for a class $\mathcal{C}$ of separable unital C*-algebras and **local $\mathcal{C}$-property** for a unital C*-algebra.

Answer 4. (1) Let A be a unital C*-algebra which is a local $\mathcal{C}$-algebra and an an action α of a finite group G. Suppose that α has the Rokhlin property, then the crossed product algebra $A \rtimes_\alpha G$ is a unital local $\mathcal{C}$-algebra.

(2) Moreover, we introduce the Rokhlin property for a conditional expectation for a pair of unital C*-algebras $A \supset P$ and show that

(a) if A is strongly self-absorbing and semiprojective, then P is strongly self-absorbing.

(b) if A is a unital local $\mathcal{C}$-algebra, then so is P.

Note that if $\mathcal{C}$ is the set of all separable, unital, $\mathcal{D}$-absorbing C*-algebras, then $\mathcal{C}$ is finitely saturated.

Local $\mathcal{C}$-property

Definition 5. (Osaka-Phillips 07) Let $\mathcal{C}$ be a class of separable unital C*-algebras. Then $\mathcal{C}$ is *finitely saturated* if the following closure conditions hold:

1. If $A \in \mathcal{C}$ and $B \cong A$, then $B \in \mathcal{C}$.

2. If $A_1, A_2, \ldots, A_n \in \mathcal{C}$ then $\bigoplus_{k=1}^n A_k \in \mathcal{C}$.

3. If $A \in \mathcal{C}$ and $n \in \mathbf{N}$, then $M_n(A) \in \mathcal{C}$.

4. If $A \in \mathcal{C}$ and $p \in A$ is a nonzero projection, then $pAp \in \mathcal{C}$.

Moreover, the *finite saturation* of a class $\mathcal{C}$ is the smallest finitely saturated class which contains $\mathcal{C}$.

Example 6. 1. Let $\mathcal{C}$ be the set of all unital C*-algebras such as $\oplus_{i=1}^n P_i M_{n_i}(C(X_i))P_i$, where P_1 is a projection in $M_{n_i}(C(X_i))$. If all X_i is a point $\{\cdot\}$, or an interval $[0,1]$, or a torus S^1. Then $\mathcal{C}$ is finitely saturated.

2. Let $\mathcal{C}$ be the set of unital C*-algebras with stable rank one. Then $\mathcal{C}$ is finitely saturated.

3. Let $\mathcal{C}$ be the set of unital C*-algebras with real rank zero. Then $\mathcal{C}$ is finitely saturated.

4. Let $\mathcal{C}$ be the set of all separable, unital, $\mathcal{D}$-absorbing C*-algebras. Then $\mathcal{C}$ is finitely saturated.

Definition 7. (Osaka-Phillips 07) Let $\mathcal{C}$ be a class of separable unital C*-algebras. A *unital local $\mathcal{C}$-algebra* is a separable unital C*-algebra A such that for every finite set $S \subset A$ and every $\varepsilon > 0$, there is a C*-algebra B in the finite saturation of $\mathcal{C}$ and a unital *-homomorphism $\varphi\colon B \to A$ (not necessarily injective) such that $\operatorname{dist}(a, \varphi(B)) < \varepsilon$ for all $a \in S$.

Rokhlin property for an inclusion of unital C*-algebras

Let A be a C*-algebra. Then we define

$$c_0(A) = \{(a_n) \in \ell^\infty(\mathbf{N}, A) \mid \lim_{n\to\infty} \|a_n\| = 0\}$$

and

$$A^\infty = \ell^\infty(\mathbf{N}, A)/c_0(A).$$

Definition 8 (Izumi 04)**.** Let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if there are mutually orthogonal projections $e_g \in A^\infty$ for $g \in G$ such that:

1. $\alpha_g^\infty(e_h) = e_{gh}$ for all $g, h \in G$.

2. $e_g a = a e_g$ for all $g \in G$ and all $a \in A$.

3. $\sum_{g\in G} e_g = 1$.

Example 9. Let $\mathbb{M}_{n^\infty} = \otimes_{k=1}^\infty \mathbb{M}_n(\mathbf{C})$ and

$$\alpha = \otimes_{k=1}^\infty \mathrm{Ad} \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ \vdots & \cdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $\{\lambda_i\}_{i=1}^n$ is the root of the unit. Then α be the automorphism of order n on $\mathbb{M}_{n^\infty}$, and α has the Rokhlin property. □

More general, let G be a finite group, λ be the left regular representation of G. We identify $B(\ell^2(G))$ with $M_{|G|}$ and consider an action of G on $M_{|G|^\infty}$ by

$$\mu_g^G = \otimes_{n=1}^\infty \mathrm{Ad}(\lambda(g)),\ g \in G.$$

Then μ^G has the Rokhlin property.

Proposition 10 (Phillips 06). Let D be an infinite tensor product C*-algebra and let $\alpha \in \mathrm{Aut}(D)$ be an automorphism of oreder 2, of the form

$$D = \otimes_{n=1}^{\infty} \mathbb{M}_{k(n)}(\mathbf{C}) \text{ and } \alpha = \otimes_{n=1}^{\infty} \mathrm{Ad}(p_n - q_n),$$

with $k(n) \in \mathbf{N}$ and where $p_n, q_n \in \mathbb{M}_{k(n)}(\mathbf{C})$ are projections with $p_n + q_n = 1$ and $\mathrm{rank}(p_n) \geq \mathrm{rank}(q_n)$ for all $n \in \mathbf{N}$. Set

$$\lambda_n = \frac{\mathrm{rank}(p_n) - \mathrm{rank}(q_n)}{\mathrm{rank}(p_n) + \mathrm{rank}(q_n)}$$

for $n \in \mathbf{N}$ and, for $m \leq n$ $\Lambda(m,n) = \lambda_{m+1}\lambda_{m+2}\cdots\lambda_n$ and $\Lambda(m,\infty) = \lim_{n\to\infty} \Lambda(m,n)$. Then the followings are equivalent:

(1) The action α has the Roklin property.

(2) There are infinitely many $n \in \mathbf{N}$ such that $\mathrm{rank}(p_n) = \mathrm{rank}(q_n)$, i.e. $\lambda_n = 0$.

(3) $D \rtimes_\alpha \mathbf{Z}_2$ is a UHF algebra.

□

Remark 11. A crossed product algebra $M_{|G|^\infty} \rtimes_{\mu^G} G$ is also an UHF algebra.

We also could construct an action which does not have the Rokhlin property.

Proposition 12 (Phillips 06)**.** Let $\alpha \in \mathrm{Aut}(D)$ be a product type automorphism of order 2 as in Proposition 10. Then the followings are equivalent:

(1) The action α has the tracial Rokhlin property.

(2) $\Lambda(m, \infty) = 0$ for all m.

□

The following observation is our motivation to introduce the Rokhlin property for the inclusion of unital C*-algebras with a finite C*-index.

Proposition 13. (Kodaka-Osaka-Teruya 08) Let α be an action of a finite group G on a unital C^*-algebra A and E the canonical conditional expectation from A onto the fixed point algebra $P = A^\alpha$ defined by

$$E(x) = \frac{1}{|G|}\sum_{g\in G}\alpha_g(x) \quad \text{for } x \in A,$$

where $|G|$ is the order of G. Then α has the Rokhlin property if and only if there is a projection $e \in A' \cap A^\infty$ such that $E^\infty(e) = \frac{1}{|G|}\cdot 1$, where E^∞ is the conditional expectation from A^∞ onto P^∞ induced by E.

Definition 14. (Kodaka-Osaka-Teruya 08) A conditional expectation E of a unital C^*-algebra A with a finite index is said to have the *Rokhlin property* if there exists a projection $e \in A' \cap A^\infty$ satisfying

$$E^\infty(e) = (\mathrm{Index}E)^{-1}\cdot 1$$

and a map $A \ni x \mapsto xe$ is injective. We call e a Rokhlin projection.

When α is an action of a finite group G on A and is satrated (i.e. $A \rtimes G = span\{xey \mid x, y \in A\}$), let P denotes the fixed point algebra A^α. We know that the canonical conditional expectation $E\colon A \to A^\alpha$ is of a finite index and we have the following basic construction :

$$A^\alpha \subset A \subset A \rtimes_\alpha G.$$

Remark 15. Let α be an action of a finite group G on a unital C^*-algebra A and E the canonical conditional expectation from A onto the fixed point algebra $P = A^\alpha$. Then α is outer. Hence E is of a finite index with $\mathrm{Index} E = |G|$. That is, there is a quasi-basis $\{(w_i, w_i^*)\}_{i=1}^n \subset A \times A$ such that

1. for any $x \in A$

$$x = \sum_{i=1}^n E(xw_i)w_i^* = \sum_{i=1}^n w_i E(w_i^* x)$$

2. $\sum_{i=1}^n w_i w_i^* = |G| = \mathrm{Index} E.$

The following is a key lemma to prove the main theorem

Lemma 16. (Kodaka-Osaka-Teruya 08)

Let $P \subset A$ be an inclusion of unital C*-algebras and E a conditional expectation from A onto P with a finite index. If E has the Rokhlin property with a Rokhlin projection $e \in A' \cap A^\infty$, then there is a unital linear map $\beta\colon A^\infty \to P^\infty$ such that for any $x \in A^\infty$ there exists the unique element y of P^∞ such that $xe = ye = \beta(x)e$ and $\beta(A' \cap A^\infty) \subset P' \cap P^\infty$. In particular, $\beta_{|_A}$ is a unital injective *-homomorphism and $\beta(x) = x$ for all $x \in P$.

We have

$$A \hookrightarrow A^\infty \overset{\beta}{\hookrightarrow} P^\infty.$$

Theorem 17. (Kodaka-Osaka-Teruya 08) Let $\mathcal{C}$ be any saturated class of semiprojective, separable unital C^*-algebras. Let $A \supset P$ be a finite index inclusion with the Rokhlin property. If A is a unital local $\mathcal{C}$-algebra, then P is also a unital local $\mathcal{C}$-algebra.

idea for the proof

Since A is a unital local $\mathcal{C}$-algebra, for finite set $S \subset P \subset A$ and $\varepsilon > 0$, there is a C^*-algebra Q in the finite saturation of $\mathcal{C}$ and a unital *-homomorphism $\rho : Q \to A$ such that S is within ε of an element of $\rho(Q)$.

$$\begin{array}{ccccc} & & & l^\infty(\mathbf{N}, P)/I_n & \\ & \bar{\beta} \nearrow & & \downarrow & \\ Q(\overset{\rho}{\hookrightarrow} A) & \overset{\beta}{\longrightarrow} & & P^\infty = l^\infty(\mathbf{N}, P)/\overline{\cup_n I_n} & \end{array}$$

Using the semiprojectivity of Q, we can lift the *-homomorphism β to a *-homomorphism $\bar{\beta}\colon Q \to \ell^\infty(\mathbf{N}, P)/I_n$ for some n. (Note that $c_o(P) = \overline{\cup_n I_n}$)

Take sufficient large $k \in \mathbf{N}$ such that $\beta_k\colon Q \to P$ is a *-homomorphism such that $S \subset_\varepsilon \beta_k(Q)$, where $\bar{\beta} = (\beta_k)_{k \in \mathbf{N}} + I_n$. $\square$

Corollary 18. Let $A \supset P$ be an inclusion of separable unital C^*-algebras with the Rokhlin property.

1. If A is a unital AF algebra, then P is a unital AF algebra.

2. If A is a unital AI algebra, then P is a unital AI algebra.

3. If A is a unital AT algebra, then P is a unital AT algebra.

4. If A is a unital AD algebra, then P is a unital AD algebra.

Rokhlin property and strongly self-absorbing

Proposition 19. Let $P \subset A$ be an inclusion of separable unital C*-algebras with index finite and A have approximately inner half flip. Suppose that E has the Rokhlin property and A is semiprojective. Then P has approximately inner half flip.

Remark 20. 1. Under the same condition for an inclusion of separable unital C*-algebras $P \subset A$ in Proposition 19 since P has approximately inner half flip map we know that P is nuclear and simple.

2. To deduce the simplicity of P we need only the simplicity of A and the Rokhlin condition for $E\colon A \to P$.

3. If $\mathcal{D}$ is a strongly self-absorbing inductive limit of recursive subhomogeneous algebras in the sense of Phillips [26], then $\mathcal{D}$ is either projectionless (i.e. the Jiang-Su algebra $\mathcal{Z}$) or a UHF algebra of infinite type by Toms and Winter [34, Corollary 5.10]. On the contrary, if $\mathcal{D}$ is a separable purely infinite strongly self-absorbing C*-algebra which satisfies the Universal Coefficients Theorem

(We write $\mathcal{D}$ is in the UCT class N.). Then $\mathcal{D}$ is either $\mathcal{O}_2$, $\mathcal{O}_\infty$ or a tensor product of $\mathcal{O}_\infty$ with a UHF algebra of infinite type by Toms and Winter [34, Corollary 5.2].

Definition 21. (Phillips 01) The class of *recursive subhomogeneous algebras* is the smallest class $\mathcal{R}$ of C*-algebras which is closed under isomorphism and such that

1. If X is a compact Husdorff space and $n \geq 1$, then $C(X, M_n) \in \mathcal{R}$.

2. $\mathcal{R}$ is closed under the following pull back construction: If $A \in \mathcal{R}$, if X is a compact Hausdorff space, if $X^{(0)} \subset X$ is closed, $\phi\colon A \to C(X^{(0)}, M_n)$ any unital homomorphism and $\rho\colon C(X, M_n) \to C(X^{(0)}, M_n)$ is the restrict homomorphism, then the pullback

$$A \oplus_{C(X^{(0)}, M_n)} C(X, M_n) \\ = \{(a, f) \in A \oplus C(X, M_n)\colon \phi(a) = \rho(f)\}$$

is in $\mathcal{R}$.

Theorem 22. Let $\mathcal{D}$ be $\mathcal{U}_\infty$ and let α be an action of a finite group G on $\mathcal{D}$. Suppose that α has the Rokhlin property. Then the crossed prodct $\mathcal{U}_\infty \rtimes_\alpha G$ is isomorphic to $\mathcal{U}_\infty$.

The following example implies that the Rokhlin property is essential in Theorem 22.

Example 23. Let $\mathcal{U}_\infty$ be the universal UHF algebra and $A = M_{2^\infty}$. Then $A \otimes \mathcal{U}_\infty \cong \mathcal{U}_\infty$.

Let α be an symmetry by Blackadar [1, Proposition 5.1.2]. Then $A \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}$ is not a AF algebra. We note that α has the tracial Rokhlin property by Phillips [28, Proposition 3.4], but does not have the Rokhlin property, since the crossed product algebra $A \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}$ is not AF algebra by Phillips [27, Theorem 2.2].

Then $\alpha \otimes id$ is a symmetry with the tracial Rokhlin property on $A \otimes \mathcal{U}_\infty (\cong A)$, and the crossed product algebra

$$\begin{aligned}(A \otimes \mathcal{U}_\infty) \rtimes_{\alpha \otimes id} \mathbf{Z}/2\mathbf{Z} &\cong (A \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}) \otimes \mathcal{U}_\infty \\ &\cong B \otimes \mathcal{U}_\infty,\end{aligned}$$

where B is the Bunce-Dedens algebras of type 2^∞ by [1, Proposition 5.4.1]. Note that $K_1(B \otimes \mathcal{U}_\infty) \neq 0$, that is, $B \otimes \mathcal{U}_\infty$ is not a AF algebra. Since a strongly self-absorbing inductive limit of type I with real rank zero C*-algebra is a UHF algebra of infinite type by Toms and Winter [34, Corollary 5.9], $B \otimes \mathcal{U}_\infty$ is not a strongly self-absorbing algebra. Hence there is a symmetry β with the tracial Rokhlin property on $\mathcal{U}_\infty$ such that $\mathcal{U}_\infty \rtimes_\beta \mathbf{Z}/2\mathbf{Z}$ is not strongly self-absorbing.

Theorem 24. Let $P \subset A$ be an inclusion of unital separable C*-algebras with index finite. Suppose that a conditional expectation $E\colon A \to P$ has the Rokhlin property and A is semiprojective and strongly self-absorbing. Then P is strongly self-absorbing.

Corollary 25. Let $P \subset A$ be an inclusion of unital separable C*-algebras with index finite. Suppose that a conditional expectation $E\colon A \to P$ has the Rokhlin property. Suppose that A is O_2 or O_∞. Then $P \cong A$.

Corollary 26. (Izumi 2002 [9, Theorem 4.2]) Let α be an action of a finite group G on $\mathcal{O}_2$. Suppose that α has the Rohklin property. Then we have

1. $\mathcal{O}_2^G \cong \mathcal{O}_2$.

2. The crossed product algebra $O_2 \rtimes_\alpha G \cong O_2$.

Remark 27. (Izumi 2004) From [10, Theorem 3.6] there is no non-trivial finite group action with the Rokhlin property on $\mathcal{O}_\infty$

Rokhlin property and $\mathcal{D}$-absorbing

We use the following characterization of the $\mathcal{D}$-absorbing.

Theorem 28. (Rordam 02)Let $\mathcal{D}$ be a strongly self-absorbing and A be any separable C*-algebra. A is $\mathcal{D}$-absorbing (i.e. $A \otimes \mathcal{D} \cong A$) if and only if $\mathcal{D}$ admits a unital *-homomorphism to $A' \cap M(A)^\infty$.

Using the above characterization and a basic Lemma 16 we have the following:

Theorem 29. Let $P \subset A$ be an inclusion of unital C*-algebras and E a conditional expectation from A onto P with a finite index. Suppose that $\mathcal{D}$ is a separable unital self-absorbing C*-algebra, A is a separable $\mathcal{D}$-absorbing, and E has the Rokhlin property. Then P is $\mathcal{D}$-absorbing.

Remark 30. If we replace the Rokhlin property by the tracail Rokhlin property, which is weaker than the Rokhlin property, then the $\mathcal{D}$-absorbing property fails. Indeed, Phillips constructed an symmetry α on a strongly self-absorbing UHF algebra $\mathcal{D}$ with the tracial Rokhlin property in the sense of Phillips such that $\mathcal{D} \rtimes_\alpha \mathbf{Z}/2\mathbf{Z}$ is not $\mathcal{D}$-absorbing. (See Example 4.11 of [28].)

Intermediate fixed point algebras

In this section we present an inclusion of unital C*-algebras $P \subset A$ which does not come from an action of finite group on A.

Proposition 31. Let A be a separable unital C*-algebra, α an action of a finite group G on A and $E\colon A \to A^G$ a canonical conditional expectation. Suppose that α has the Rokhlin property. Then we have

1. For any subgroup H of G the restricted E to A^H, which is a conditional expectation from A^H onto A^G, has the Rokhlin property.

2. If A is a unital local $\mathcal{C}$-algebra, then for any subgroup H of G A^H is a unital local $\mathcal{C}$-algebra.

3. Let $\mathcal{D}$ be a strongly self-absorbing C*-algebra and A be $\mathcal{D}$-absorbing. Then for any subgroup H of G A^H is $\mathcal{D}$-absorbing.

4. If $A = \mathcal{O}_2$, then for any subgroup H of G $A^H \cong \mathcal{O}_2$.

Remark 32. Let A be a unital C*-algebra and α be an action from a finite group G on A. Let H be a subgroup of G. Then the condition that an inclusion $A^G \subset A^H$ is isomorphic to $B^K \subset B$ for some C*-algebra B and an action from a finite group K on B implies that H is a normal subgroup of G (c.f. [32]). Hence from Proposition 31 we have examples of conditional expectations for inclusions of unital C*-algebras with the Rokhlin property which do not come from finite group actions.

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